JOURNAL OF ALGEBRA 55, 403–445 (1978)

A 2-Local Characterization of Janko's Simple Group J_4

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Communicated by B. Huppert

Received December 21, 1977

INTRODUCTION

In this paper J_4 denotes a finite simple group discovered by Janko and described in [13].

The group J_4 contains exactly one conjugacy class of 2-central and one class of non-2-central involutions with representatives z and t, respectively. $C_{J_4}(z)$ is isomorphic to a product of $Ex^+(2^{13})$and a group of type $\hat{6}\mathrm{Aut}(M_{22})$ with amalgamated centers, whereas $C_{J_4}(t)$ is isomorphic to a semidirect product of $E(2^{11})$ and $\mathrm{Aut}(M_{22})$. Furthermore, there is exactly one $E(2^{11})$-subgroup in an S_2-subgroup of J_4 and its normalizer is isomorphic to $E(2^{11})$ extended splittingly by the Mathieu group M_{24}.

It is also well known that there is one conjugacy class of self-centralizing elementary Abelian subgroups of order 2^{10}. If V is a representative of that class, then $N_{J_4}(V) = V \cdot L$ with $V \cap L = 1$ and $L \simeq GL_5(2)$. Moreover, L induces orbits with lengths of 155 and 868 on V and thus acts irreducibly on V. The purpose of this paper is to characterize J_4 by means of that 2-local subgroup.

MAIN THEOREM. *Let G be a finite group of order divisible by 2^{21}; furthermore, suppose that G contains an elementary Abelian subgroup V of order 2^{10} such that $N_G(V)/O(N_G(V))$ is isomorphic to an irreducible extension of $E(2^{10})$ by $GL_5(2)$. Then one of the following holds:*

(i) *The group G contains a normal series $O(G) \lhd G_0 \trianglelefteq G_1 \lhd G$ such that $|G: G_1| = 2$, $\mathrm{Syl}_2(G_0) = \mathrm{Syl}_2(G_1)$, and $G_1/G_0 \simeq O(C_{G_1}(X))/O(C_{G_0}(X))$ for any $X \in \mathrm{Syl}_2(G_0)$; $G_0/O(G)$ is a non-Abelian simple group and $N_{G_0}(V)/C_{G_0}(V) \simeq GL_5(2)$. Moreover, G_0 contains at least four conjugacy classes of involutions which are already G-classes.*

(ii) *The group $G/O(G)$ is isomorphic to J_4.*

Remark. If we are in case (i) of the main theorem and if $N_{G_0}(V \bmod O(G))/O(G)\,V$ is isomorphic to $GL_5(2)$ as well as $O(C_{G_0}(z \bmod O(G))) \leq O(G)$ for

0021-8693/78/0552–0403$02.00/0

2-central involutions z of G_0 then, by results of [4], $G_0/O(G)$ is isomorphic to the orthogonal group $D_5(2)$; hence $G/O(G) \cong \mathrm{Aut}(D_5(2))$.

It seems very likely that $G/O(G)$ is isomorphic to $\mathrm{Aut}(D_5(2))$ in any case of (i).

Outline of Proof. First of all we prove that $N_G(V)/C_G(V)$ induces orbits with lengths of 155 and 868 on V and that $N_G(V)$ splits over V. Thus $N_G(V) = V \cdot L$ with $V \cap L = 1$, $O(L) = O(N_G(V)) = O(C_G(V))$, and $L/O(L) \cong GL_5(2)$.

Inspecting $N_G(V)$ we are able to establish that an S_2-subgroup $V \cdot T$ of $N_G(V)$ contains exactly one extra-special subgroup E of order 2^{13} and exactly three elementary Abelian subgroups V, W, and W^* of order 2^{10} such that $VW = VW^* = WW^*$, $V \cap W = V \cap W^* = Z(VW) \cong E(2^6)$, and $N_G(V) \cap N_G(W) \cap N_G(W^*) = VW \cdot A$ with $A/O(A) \cong \mathrm{Alt}_8$. Moreover, V and W are completely reducible A-modules, whereas W^* is not completely reducible. By assumption there exists a 2-subgroup S of G containing VT with index two. Furthermore, there exists an involution s in $S - VT$ such that $U = \langle W^*, s\rangle$ is the only elementary Abelian subgroup of order 2^{11} in S, which is already an S_2-subgroup of G. Clearly, s acts on both VW and $VW \cdot A$.

Assuming $[A, s] \leqslant O(A)$ fusion arguments yield the statements of case (i).

So we are left with the case that $[A, s] \nleqslant O(A)$; then $C_A(s \bmod O(A))/O(A) \cong \mathrm{Alt}_7$ and $Z(VW) \leqslant [A, s] \leqslant O(A) \cdot Z(VW)$. We can easily prove that $C_G(U) = O(C_G(U)) \times U$ and $C_G(E) = O(C_G(E)) \times Z(E)$. Fusion arguments together with the main results of [8, 11] now yield $N_G(U)/C_G(U) \cong M_{24}$ and $N_G(E)/(C_G(E)\, E) \cong \hat{3}\, \mathrm{Aut}(M_{22})$; moreover, G contains exactly two conjugacy classes of involutions. Nevertheless we are not in a position to make use of Reifart's results [14]. But extended applications of Goldschmidt's Corollary 4 [6] yield that an S_2-subgroup of $C_G(j)/\langle j\rangle$, j an involution of G, contains a strongly closed elementary Abelian subgroup. This finally implies that $C_G(j)$ is 2-constrained and that $C_G(j)/O(C_G(j))$ is isomorphic to the corresponding involution centralizer in J_4. Now, by results of [15, 18], $G/O(G)$ is isomorphic to J_4.

Conventions and Standard Notation

For convenience we make the following definitions:

(0.1) Given groups A and B we write $A \underset{\sim}{\leqslant} B$ ($A \underset{\sim}{\nleqslant} B$) provided A is (is not) isomorphic to some (any) subgroup of B.

(0.2) Let H be a finite group such that H' is non-Abelian simple and $|H : H'| \leqslant 2$. The group X is said to be isomorphic to $\hat{n}H$, if X' is isomorphic to an n-fold covering of H', $X/Z(X') \cong H$, and if elements of $X - X'$ act invertingly on $Z(X')$.

(0.3) Let A be a group and $B \leqslant N(A)$; then $A \cdot B$ denotes the semidirect product of A and B and $A * B$ denotes a product of A and B with amalgamated centers, whereas AB denotes any product of A and B.

(0.4) $E(p^n)$ and $Ex(p^n)$ denote an elementary Abelian and an extra-special group of order p^n, and Z_n, D_n, and F_n denote a cyclic, dihedral, and Frobenius group of order n, respectively. Furthermore, Σ_n is the symmetric group on n letters and Alt_n designates the alternating group on n letters.

(0.5) Suppose the group N acts on a set V and induces n orbits on V with representatives $v_1, \ldots, v_n$ such that $|\,\mathrm{ccl}_N(v_i)| = m_i$, $i \in \{1, \ldots, n\}$. Then we write $(N!\ V) = m_1 v_1 + \cdots + m_n v_n$ or $(N!\ V) = (\mathbf{m}_1) + \cdots + (\mathbf{m}_n)$; moreover, $(N!\ V) = k_1(\mathbf{m}_1) + \cdots + k_t(\mathbf{m}_t)$ means that there are k_i N-orbits in V which have length m_i, $i \in \{1, \ldots, t\}$.

Assume there is a subset U of V which is acted upon by N; if $(N!\ V - U) = m_1 v_1 + \cdots + m_n v_n$ and if the action of N on U is well known, then we also write $(N!\ V) = U + m_1 v_1 + \cdots + m_n v_n$.

Finally, if V is a group, we also write $(N!\ V)$ instead of $(N!\ V^{\#})$.

(0.6) Given elements $x, y_1, y_2, y_3, \ldots$ then $x\colon y_1 \to y_2 \to y_3 \cdots$ is equivalent to $y_1^x = y_2$, $y_2^x = y_3$, etc.

(0.7) Given integers m and n then $m \mid n$ ($m \nmid n$) means that m does (does not) divide n.

All other notation used is standard and may be found in [7] or [12].

I. THE GROUP $GL_5(2)$

1. Structure of the Group $GL_5(2)$

Let L denote the general linear group of 5×5 matrices over $GF(2)$. Then L is a simple group of order $2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$.

For convenience we introduce the following notation for some elements of L:

$$r_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$r_1 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 1&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix},\quad d_1 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 1&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix},\quad t_1 = \begin{pmatrix} 1&0&0&0&0\\ 1&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix},$$

$$r_2 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&1&0&0&1 \end{pmatrix},\quad d_2 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&1&0&1 \end{pmatrix},\quad t_2 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&1&1 \end{pmatrix},$$

$$r_3 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&1&0&1&0\\ 0&0&0&0&1 \end{pmatrix},\quad d_{1+} = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&1&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix},\quad d_{2+} = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&1&1&0\\ 0&0&0&0&1 \end{pmatrix},$$

$$a_1 = \begin{pmatrix} 1&1&0&0&0\\ 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix},\quad a_2 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&1\\ 0&0&0&1&0 \end{pmatrix},\quad a_3 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&1&0\\ 0&0&1&0&0\\ 0&0&0&0&1 \end{pmatrix};$$

$$q_0 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&1&0\\ 0&0&0&1&0\\ 0&1&1&1&0\\ 0&0&0&0&1 \end{pmatrix},\quad q_1 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&1&0\\ 0&0&0&0&1\\ 0&0&1&0&0 \end{pmatrix},\quad q_2 = \begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 1&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix};$$

$$d = \begin{pmatrix} 1&1&0&0&0\\ 0&1&1&0&0\\ 0&0&1&1&0\\ 0&0&0&1&1\\ 1&1&1&1&1 \end{pmatrix};\quad f = \begin{pmatrix} 0&1&1&0&0\\ 0&0&0&1&1\\ 1&0&0&0&1\\ 0&1&0&1&1\\ 1&0&0&1&1 \end{pmatrix},\quad f_1 = \begin{pmatrix} 1&1&0&1&0\\ 1&0&0&0&1\\ 0&0&1&0&0\\ 0&1&0&1&0\\ 1&1&0&0&1 \end{pmatrix};$$

$$a = \begin{pmatrix} 0&0&1&0&0\\ 1&0&1&1&0\\ 1&0&1&0&0\\ 0&1&1&1&0\\ 0&0&0&0&1 \end{pmatrix},\quad b_1 = \begin{pmatrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&1 \end{pmatrix},\quad b_2 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&1&0&0\\ 0&0&1&0&0\\ 1&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix},$$

$$y = \begin{pmatrix} 1&0&1&0&0\\ 0&1&0&1&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \qquad t_0 = \begin{pmatrix} 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&1&0&0\\ 1&0&0&0&0\\ 0&1&0&0&0 \end{pmatrix}.$$

Hence, $r_0^2 = r_1^2 = \cdots = d_{1+}^2 = d_{2+}^2 = 1$, $a_1^3 = a_2^3 = a_3^3 = 1$, $q_0^7 = q_1^7 = q_2^7 = d^{31} = f^5 = f_1^5 = 1$, $d^f = d^2$, $q_1^{a_2} = q_1^2$, $q_1^{a_1} = q_1$, and $(a_1a_2)^{f_1} = a_1a_2$.

Furthermore, $[r_i, r_j] = [r_i, d_k] = [r_i, d_{k+}] = [r_k, t_k] = [d_k, d_{k+}] = [d_k, t_k] = [r_0, t_k] = [t_k, d_l] = [t_k, d_{l+}] = [t_1, t_2] = 1$, $[t_k, d_{k+}] = d_k$, $[d_{1+}, d_{2+}] = r_3$, $[t_k, r_3] = [d_k, d_{l+}] = r_k$, and $[t_k, r_l] = [d_1, d_2] = r_0$ for integers i, j, k, l with $\{i, j\} \subset \{0, 1, 2, 3\}$ and $\{k, l\} = \{1, 2\}$.

Finally we put $T = \langle r_i, d_j, d_{j+}, t_j \mid 0 \leqslant i \leqslant 3, 1 \leqslant j \leqslant 2\rangle$, $E_i = \langle r_0, r_i, d_i, t_i\rangle$, and $R_i = \langle r_0, r_1, r_2, r_3, d_i, d_{i+}\rangle$ for $i \in \{1, 2\}$,

$$K_1 = \left\{ \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&&&\\ 0&0&&*&\\ 0&0&&& \end{pmatrix} \middle| * \in GL_3(2) \right\},$$

$$K_2 = \left\{ \begin{pmatrix} &&&0&0\\ &*&&0&0\\ &&&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix} \middle| * \in GL_3(2) \right\},$$

$$A_1 = \left\{ \begin{pmatrix} 1&0&0&0&0\\ 0&&&&\\ 0&&*&&\\ 0&&&&\\ 0&&&& \end{pmatrix} \middle| * \in GL_4(2) \right\},$$

$$A_2 = \left\{ \begin{pmatrix} &&&&0\\ &&&&0\\ &&*&&0\\ &&&&0\\ 0&0&0&0&1 \end{pmatrix} \middle| * \in GL_4(2) \right\},$$

$A_0 = A_1 \cap A_2$, $H_0 = C_L(r_0)$, and $H_1 = C_L(r_1r_2)$.

Then T is an S_2-subgroup of L, $T' = Z_3(T) = \langle r_0, r_1, r_2, r_3, d_1, d_2\rangle$, $Z_2(T) = \langle r_0, r_1, r_2\rangle$, and $T'' = Z(T) = \langle r_0\rangle$; the groups E_1, E_2, R_1, and R_2 are the only maximal Abelian normal subgroups of T and the group E_1E_2 is the only extra-special subgroup of order 2^7 in T.

Suppose $i \in \{1, 2\}$; then $C_L(E_i) = E_i \cong E(2^4)$, $N_L(E_i) = E_i \cdot A_i$, $C_L(R_i) = R_i \cong E(2^6)$, and $N_L(R_i) = R_i \cdot (K_i \times \langle a_i, t_i \rangle)$ with $\langle a_i, t_i \rangle \cong \Sigma_3$.

The elements r_0 and $r_1 r_2$ are representatives of the two conjugacy classes of involutions of L; we have $H_0 = E_1 E_2 \cdot A_0$ and $H_1 = R_1 R_2 \cdot \langle a_1 a_2, t_1 t_2 \rangle$ with $\langle a_1 a_2, t_1 t_2 \rangle \cong \Sigma_3$.

Moreover, $\langle a_1, a_2 \rangle \in \mathrm{Syl}_3(L)$ with $N_L(\langle a_1, a_2 \rangle) \cong E(3^2) \cdot D_8$, $C_L(a_1) = \langle a_1 \rangle \times K_1 \cong Z_3 \times GL_3(2)$, $C_L(a_1 a_2) = \langle a_1 a_2 \rangle \times \langle r_1 r_2, r_0 r_1 r_3, a_1 a_2^2, f_1 \rangle \cong Z_3 \times \mathrm{Alt}_5$, $N_L(\langle q_0 \rangle) = \langle r_0 r_3^{t_0}, r_0 \rangle \times \langle q_0, a_3^2 r_1 b_2 \rangle \cong \Sigma_3 \times F_{21}$, $N_L(\langle f \rangle) \cong (Z_3 \times Z_5) \cdot Z_4$, and $N_L(\langle d \rangle) = \langle d, f \rangle \cong Z_{31} \cdot Z_5$.

(1.1) Put $A = \langle a_1, a_3, b_1 t_1, b_2 t_1, a \rangle$ and $B = \langle a_1, a_3, b_1, b_2, t_1 \rangle$. Then $B' = \langle a_1, a_3, b_1 t_1, b_2 t_1 \rangle \cong \mathrm{Alt}_6$, $B = B' \cdot \langle t_1 \rangle \cong \Sigma_6$, and $A = \langle B', a \rangle = \langle B', y \rangle \cong \mathrm{Alt}_7$.

Furthermore, Alt_8 contains exactly one conjugacy class of subgroups isomorphic to each of Σ_6, Alt_6, and Alt_7.

Proof. Put $S_3 = \langle a_1, a_3 \rangle$ and let X and Y be subgroups of A_2 ($\cong \mathrm{Alt}_8$) such that $X \cong \Sigma_6$, $Y \cong \mathrm{Alt}_7$, and $S_3 \leqslant X \cap Y$. As $N_X(S_3) = N_L(S_3) = S_3 \cdot \langle t_1, d_{2+}, b_1 \rangle \cong E(3^2) \cdot D_8$, $N_{X'}(S_3) = N_Y(S_3) = S_3 \cdot \langle b_1 t_1 \rangle \cong E(3^2) \cdot Z_4$ with $(b_1 t_1)^2 = t_1 d_{2+}$, and S_2-subgroups of both, X' and Y are isomorphic to D_8, we compute that $\{x \in A_2 \mid x^2 = 1, (b_1 t_1)^x = (b_1 t_1)^{-1}\} = t_1 \langle b_2, b_1 t_1 \rangle$. Thus $\langle b_1 t_1, b_2 t_1 \rangle \in \mathrm{Syl}_2(X') \cap \mathrm{Syl}_2(Y)$, because both, X' and Y contain exactly one class of involutions only. Since $N_{X'}(S_3)$ is a maximal subgroup of X' and $N_X(S_3)$ is a maximal subgroup of X, we get $X' = \langle N_{X'}(S_3), b_2 t_1 \rangle = \langle S_3, b_1 t_1, b_2 t_1 \rangle \leqslant Y$ and $X = \langle N_X(S_3), b_2 t_1 \rangle = \langle S_3, \langle b_1 t_1, b_2 t_1 \rangle \times \langle b_2 \rangle \rangle = X' \cdot \langle t_1 \rangle$.

It is well known that $|C_{X'}(t_1 d_{2+})| = 2^3$ and $C_A(t_1 d_{2+}) = (E \times \langle u \rangle) \langle b_1 t_1 \rangle$ with $E \cong E(2^2)$, $o(u) = 3$, $(b_1 t_1)^2 = t_1 d_{2+} \in E$ and $u^{b_1 t_1} = u^2$; inspecting $C_{A_2}(t_1 d_{2+})$ we get $u \in \{a, a^{t_1}\}$. Since $t_1 \in N_{A_2}(X')$, we have $Y = \langle X', a \rangle$ or $Y = \langle X', a \rangle^{t_1}$. As $y = a^2 b_2 t_1$, the statements above have been proved.

(1.2) Each of L and $E_2 A_2$ has exactly one conjugacy class of subgroups isomorphic to Alt_8.

Proof. Let X be a subgroup of L or $E_2 \cdot A_2$ isomorphic to Alt_8. Without loss of generality we may assume that $S_3 = \langle a_1, a_3 \rangle$ is contained in X. So $X \geqslant N_L(S_3) = S_3 \cdot \langle t_1, d_{2+}, b_1 \rangle \cong E(3^2) \cdot D_8$ and a_1 and $a_1 a_3$ are representatives of the two conjugacy classes of elements of order three in X. Clearly, X contains $N_L(\langle a_1 a_3 \rangle)$ which is isomorphic to $(Z_3 \times \mathrm{Alt}_5) \cdot Z_2$. Hence, $X_0 = \langle N_L(S_3), N_L(\langle a_1 a_3 \rangle) \rangle \leqslant X$ and $|X_0| = 2^{3+\alpha} \cdot 3^2 \cdot 5 \cdot 7^\beta$ with $0 \leqslant \alpha \leqslant 3$, $0 \leqslant \beta \leqslant 1$, and $\alpha + \beta \geqslant 1$.

If F is an S_5-subgroup of $N_L(\langle a_1 a_3 \rangle)$, then $(Z_3 \times Z_5) \cdot Z_4 \cong N_L(F) \leqslant N_L(\langle a_1 a_3 \rangle) \leqslant X_0$. Thus, by a theorem of Sylow, $2^\alpha \cdot 7^\beta \equiv 1 \mod 5$; this immediately implies $\beta = 1$ and $\alpha = 3$. So $X_0 = X$ and the claimed results hold.

TABLE I

Conjugacy Classes of L

Elt. x	$o(x)$	x^2	$\|C_L(x)\|$	$C_L(x)$
1	1		$\|L\|$	L
r_0	2		$2^{10} \cdot 3 \cdot 7$	H_0
r_1r_2	2		$2^9 \cdot 3$	H_1
$r_1r_3t_1t_2$	4	r_1r_2	2^5	$\langle r_1r_3t_1t_2\rangle \times \langle r_0, d_1, d_2\rangle$
d_1d_2	4	r_0	$2^7 \cdot 3$	$\langle r_0, r_1, r_2, d_1d_2, t_1, t_2, d_{1+}d_{2+}q_0^3, r_3\rangle$
$r_3d_1d_2$	4	r_0	2^7	$(R_1 \cap R_2) \cdot \langle d_1d_2, t_1d_{2+}, t_2d_{1+}\rangle$
$t_1d_{1+}d_{2+}t_2$	8	$r_3d_1d_2$	2^4	$\langle t_1d_{1+}d_{2+}t_2\rangle \times \langle r_1r_2\rangle$
a_1a_2	3		$2^2 \cdot 3^2 \cdot 5$	$\langle a_1a_2\rangle \times \langle r_1r_2, r_0r_1r_3, f_1\rangle$
a_1	3		$2^3 \cdot 3^2 \cdot 7$	$\langle a_1\rangle \times K_1$
f_1	5		$3 \cdot 5$	$\langle a_1a_2\rangle \times \langle f_1\rangle$
q_0	7		$2 \cdot 3 \cdot 7$	$\langle r_3^{t_0}, r_0\rangle \times \langle q_0\rangle$
q_0^{-1}	7		$2 \cdot 3 \cdot 7$	
d	31		31	$\langle d\rangle$
d^3	.		.	.
d^5	.		.	.
d^7	.		.	.
d^{11}	.		.	.
d^{30}	31		31	$\langle d\rangle$
$r_1r_2a_1a_2$	6	$(a_1a_2)^2$	$2^2 \cdot 3$	$\langle r_1r_2, r_0r_1r_3\rangle \times \langle a_1a_2\rangle$
d_2a_1	6	a_1^2	$2^3 \cdot 3$	$\langle d_2, d_{2+}, t_2\rangle \times \langle a_1\rangle$
$d_{2+}t_2a_1$	12	$d_2a_1^2$	$2^2 \cdot 3$	$\langle d_{2+}t_2a_1\rangle$
r_0q_0	14		$2 \cdot 7$	$\langle r_0q_0\rangle$
$r_0q_0^{-1}$	14		$2 \cdot 7$	$\langle r_0q_0\rangle$
$a_1a_2f_1$	15		$3 \cdot 5$	$\langle a_1a_2f_1\rangle$
$a_1^2a_2^2f_1$	15		$3 \cdot 5$	$\langle a_1a_2f_1\rangle$
a_1q_1	21		$3 \cdot 7$	$\langle a_1q_1\rangle$
$a_1q_1^6$	21		$3 \cdot 7$	$\langle a_1q_1\rangle$

(1.3) Making use of fairly standard arguments it is easy to verify the statements listed in Table I.

(1.4) Let X be a subgroup of L such that $|L: X| \leqslant 2^{10}$. Then X is conjugate to one of the groups listed below, where $i \in \{1, 2\}$, $A^{(i)} \simeq \mathrm{Alt}_7$, and $B^{(i)} \simeq \Sigma_6$.

Conjugate of X	$\mid X \mid$	$\mid L: X \mid$
A_1	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$2^4 \cdot 31 = 496$
$E_i \cdot A^{(i)}$	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	$2^3 \cdot 31 = 248$
$E_i \cdot B^{(i)}$	$2^8 \cdot 3^2 \cdot 5$	$2^2 \cdot 7 \cdot 31 = 868$
$R_i \cdot K_i$	$2^9 \cdot 3 \cdot 7$	$2 \cdot 3 \cdot 5 \cdot 31 = 930$
$R_i \cdot (K_i \times \langle a_i \rangle)$	$2^9 \cdot 3^2 \cdot 7$	$2 \cdot 5 \cdot 31 = 310$
$R_i \cdot (K_i \times \langle t_i \rangle)$, H_0	$2^{10} \cdot 3 \cdot 7$	$3 \cdot 5 \cdot 31 = 465$
$N_L(R_i)$	$2^{10} \cdot 3^2 \cdot 7$	$5 \cdot 31 = 155$
$N_L(E_i)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7$	31
L	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$	1

(1.5) Suppose X is a subgroup of L of order divisible by 31. Then $X = L$ or $X \leqslant N_L(X) \cong Z_{31} \cdot Z_5$.

2. 10-Dimensional $GF(2)$-Representations of $GL_5(2)$

Let N denote a finite group containing an elementary Abelian normal subgroup V of order 2^{10} such that $N/V \cong GL_5(2)$ and $C_N(V) = V$. Furthermore, let N_2 be an S_2-subgroup of N and let f, q, and d be elements of N of orders 5, 7, and 31, respectively.

(2.1) (a) Either $\mid C_V(f) \mid = 2^6$, $\mid C_V(d) \mid = \mid Z(N) \mid = 2^5$, and $\mid C_V(q) \mid = 2^7$ or $\mid C_V(f) \mid = 2^2$, $\mid C_V(d) \mid = \mid Z(N) \mid = 1$, and $\mid C_V(q) \mid \in \{2, 2^4\}$.

(b) $N_N(N_2) = N_2$ and $Z(N_2) \cap \mathrm{ccl}_N(z) = \{z\}$ for any $z \in Z(N_2)$.

(c) Let U be a subgroup of N containing V such that $\mid N: U \mid \in \{31, 155, 248, 868, 310, 930\}$; then $N_N(U)/V \cong E(2^6) \cdot (L_3(2) \times \Sigma_3)$ if $\mid N: U \mid \in \{310, 930\}$ and $N_N(U) = U$ otherwise.

Proof. Trivial.

(2.2) The group N acts on V in one of the following ways:

$(N! \, V)$	$\mid Z(N_2) \mid$	$\mid C_V(f) \mid$	$\mid C_V(q) \mid$
(930) + 3**(31)**	2^2	2^2	2^4
(868) + **(155)**	2	2^2	2
2**(496)** + **(31)**	2	2^2	2^4
2**(496)** + 31**(1)**	2^5	2^6	2^7
(496) + **(465)** + 2**(31)**	2^2	2^2	2^4
32**(31)** + 31**(1)**	2^5	2^6	2^7

Proof. Make use of (1.4), (2.1), and the fact that $N_N(P \bmod V)$ controls fusion in $C_V(P)$ for any Sylow subgroup P of N.

(2.3) If $(N!\,V) = (\mathbf{496}) + (\mathbf{465}) + 2(\mathbf{31})$ or $(N!\,V) = (\mathbf{930}) + 3(\mathbf{31})$, then N acts reducibly on V.

Proof. Suppose that N satisfies the assumptions above. As d has no fixed point in $V^{\#}$, we may assume that $N_N(\langle d\rangle) = \langle d, f\rangle$ with $d^f = d^2$. Since $|C_V(f)| = 2^2$, we get $(\langle d, f\rangle!\,V) = 3(\mathbf{31}) + 6(\mathbf{155})$.

Without loss of generality we regard $\bar{N} = N/V$ as a subgroup of $\mathscr{L} = GL_{10}(2)$. Assuming I, D, and F to be elements of $GL_5(2)$ of order 1, 31, and 5, respectively, with $D^F = D^2$ we put $x_1 = \left(\begin{smallmatrix} D & 0 \\ 0 & I\end{smallmatrix}\right)$, $x_2 = \left(\begin{smallmatrix} I & 0 \\ 0 & D\end{smallmatrix}\right)$, $y_1 = \left(\begin{smallmatrix} F & 0 \\ 0 & I\end{smallmatrix}\right)$, and $y_2 = \left(\begin{smallmatrix} I & 0 \\ 0 & F\end{smallmatrix}\right)$; furthermore let $i \in \{1, 2\}$. Then $(x_i)^{y_i} = x_i^2$, $[x_1, x_2] = [y_1, y_2] = 1$, $\langle x_1, x_2\rangle \in \mathrm{Syl}_{31}(\mathscr{L})$, and $\langle y_1, y_2\rangle \in \mathrm{Syl}_5(\mathscr{L})$. Since $\bar{d}$ has no fixed points in $V^{\#}$, we may assume that $\bar{d} = x_1^{\,l}x_2^{\,k}$ with suitable integers k, $l \in \{1, 2,\ldots, 30\}$. As $5 \nmid |C_{\mathscr{L}}(\bar{d})|$ and $x^{y_1y_2} = x^2$ for any $x \in \langle x_1, x_2\rangle$, $\langle y_1y_2\rangle = \langle \bar{f}\rangle \in \mathrm{Syl}_5(N_{\mathscr{L}}(\langle \bar{d}\rangle))$. Furthermore, $\langle \bar{d}, \bar{f}\rangle \leqslant N_{\mathscr{L}}(\langle x_i\rangle)$ and thus $C_V(x_i) = [C_V(x_i), \langle \bar{d}, \bar{f}\rangle] \cong E(2^5)$. Since there are at least two N-orbits of length 31 in V, $\bar{N}$ acts on $C_V(x_1)$ or on $C_V(x_2)$. This proves the statement above.

(2.4) If $(N!\,V) = (\mathbf{31}) + 2(\mathbf{496})$, then N acts reducibly on V.

Proof. Suppose $(N!\,V) = 31v_1 + 496v_2 + 496v_3$; then $C_N(v_1)/V \cong E(2^4) \cdot \mathrm{Alt}_8$ and $C_N(v_j)/V \cong \mathrm{Alt}_8$ for $j \in \{2, 3\}$. Put $C_1 = C_N(v_1)$, $C_2 = C_N(v_2)$, and $Q = O_2(C_1)$.

Since $GL_5(2)$ contains exactly one class of subgroups isomorphic to Alt_8, we may assume that $C_N(v_3) = C_2$ and $C_1 = QC_2 > N_2$ with $Q \cap C_2 = V$ and $Z(N_2) = \langle v_1\rangle$. Clearly, $V > Q' = \langle x^2 \mid x \in Q\rangle = \Phi(Q) > \langle v_1\rangle$; moreover, if $Q' = \langle v_1\rangle$, then $Z(Q) > \langle v_1\rangle$, because $|Q| = 2^{14}$ and thus Q is not extra-special.

As $C_V(C_2) = \{1, v_1, v_2, v_3\}$, there exists a C_1-invariant subgroup U of V such that $\langle v_1\rangle < U < V$ and $|U| \in \{2^5, 2^6, 2^9\}$. Let $j \in \{2, 3\}$ and $w_j \in \mathrm{ccl}_N(v_j) - \mathrm{ccl}_{C_1}(v_j)$; then $|\mathrm{ccl}_{C_1}(v_j)| = 16$ and $80 \mid |\mathrm{ccl}_{C_1}(w_j)|$. Since $2^5 = 1 + 31$, $2^6 = 1 + 31 + 2 \cdot 16$, and $2^9 = 1 + 31 + 6 \cdot 80$, we easily conclude that $\mathrm{ccl}_N(v_1)$ is contained in U. Thus, N acts reducibly on V.

Now suppose that N acts irreducibly on V. Hence, by the results obtained so far, $(N!\,V) = 155v_1 + 868v_2$.

As we are going to determine $C_N(v_i)/V$, $i \in \{1, 2\}$, we may assume that $N = V \cdot L$ with $L \cong GL_5(2)$. Furthermore, we use notation introduced in Section 1. Thus, w.l.o.g. $C_L(v_1) = N_L(R_i)$ and $C_L(v_2) = E_j \cdot B_j$ with $\Sigma_6 \cong B_j < A_j$ and suitable $i, j \in \{1, 2\}$.

If $i = j$, then $\langle v_1, v_2\rangle \leqslant C_V(E_i)$, which is A_i-invariant. Since $|A_i \cap N_L(R_i)| = 2^6 \cdot 3 \cdot 7$ and $|A_i : B_i| = 28$, we get $2^6 \mid |C_V(E_i)| \mid 2^9$. But this finally yields a contradiction, because elements of order 5 or 7 of A_i do not act fixed-point-freely on $C_V(E_i)^{\#}$. Therefore $i \neq j$.

Since R_1 and R_2 as well as E_1 and E_2 occur symmetrically in the S_2-subgroup T of L, we may and do assume that $(i, j) = (1, 2)$ and $B_2 = B$. Thus, $C_L(v_1) = N_L(R_1)$ and $C_L(v_2) = E_2 \cdot B$.

We have proved that there are at most two nonisomorphic irreducible $GL_5(2)$-modules of dimension 10 over $GF(2)$.

Now let W denote a standard $SL_5(2)$-module of dimension 5 over $GF(2)$. Then it is well known that $\Lambda^i(W)$, $0 \leqslant i \leqslant 4$, are pairwise nonisomorphic irreducible $SL_5(2)$-modules over $GF(2)$ with $\dim \Lambda^i(W) = \binom{5}{i}$. As $\binom{5}{2} = \binom{5}{3} = 10$ and $GL_5(2) = SL_5(2)$, application of [2, Theorem 4.1] to N/V and V yields the following theorem.

THEOREM 1. *Let N be a finite group containing an elementary Abelian normal subgroup V of order 2^{10} such that $N/V \cong GL_5(2)$. If N/V acts irreducibly on V, then N splits over V and induces orbits of length* 155 *and* 868 *on V.*

II. A CHARACTERIZATION OF J_4

In this part, G designates a finite group satisfying the following assumptions:

(A1) 2^{21} divides $|G|$.

(A2) G contains an elementary Abelian subgroup V of order 2^{10} such that $N_G(V)/(O(N_G(V)) \times V) \cong GL_5(2)$ and $N_G(V)$ acts irreducibly on V.

Thus, by results of Part I, $N_G(V) = V \cdot L$ with $V \cap L = 1$, $O(L) = O(N_G(V)) = O(C_G(V))$, and $L/O(L) \cong GL_5(2)$. Moreover, $C_G(V) = O(L) \times V$.

In what follows $r_0, r_1, r_2, r_3, d_1, d_{1+}, t_1, d_2, d_{2+}, t_2$, and a_1, a_2, a_3, and q_0, q_1, q_2, and d, f, f_1, and a, b_1, b_2, y, and t_0 denote elements of $L - O(L)$ such that their images in $L/O(L)$ correspond to those 5×5 matrices over $GF(2)$ labeled with the same letters as in Chapter I.

Since S_2-subgroups of L and $L/O(L)$ are isomorphic, we may assume that $r_j^2 = d_j^2 = d_{i+}^2 = t_i^2 = 1$ for $j \in \{0, 1, 2, 3\}$ and $i \in \{1, 2\}$ and that these involutions generate an S_2-subgroup T of L.

Moreover, for $i \in \{1, 2\}$, let $E_i = \langle r_0, r_i, d_i, t_i \rangle$ and $R_i = \langle r_0, r_1, r_2, r_3, d_i, d_{i+} \rangle$. Then $E_i \cong E(2^4)$, $N_L(E_i) = E_i \cdot A_i$ with $O(A_i) = O(N_L(E_i)) = O(C_L(E_i))$ and $A_i/O(A_i) \cong \mathrm{Alt}_8$, and $C_L(E_i) = O(A_i) \times E_i$; furthermore, $R_i \cong E(2^6)$, $C_L(R_i) = O(N_L(R_i)) \times R_i$ and $N_L(R_i) = C_L(R_i)\, K_i \langle a_i, t_i \rangle$ with $\langle a_i, t_i \rangle / O(\langle a_i, t_i \rangle \cong \Sigma_3$, $K_i/O(K_i) \cong GL_3(2)$, and $[K_i, \langle a_i, t_i \rangle] \leqslant K_i \cap \langle a_i, t_i \rangle \leqslant O(K_i) \cap O(\langle a_i, r_i \rangle) \leqslant O(N_L(R_i)) \leqslant O(L)$.

Let H_0 and H_1 denote the centralizers of r_0 and $r_1 r_2$ in L, respectively; then $H_1 = O(H_1)\, R_1 R_2 \langle a_1 a_2, t_1 t_2 \rangle$ and $H_0 = O(H_0)\, E_1 E_2 A_0$ with $O(A_0) \leqslant O(H_0)$ and $A_0/O(A_0) \cong GL_3(2)$.

We may assume that the following hold:

$A_0 = \langle r_3, d_{1+}, d_{2+}, a_3, q_0 \rangle$, $A_1 = O(A_1) \langle r_2, d_2, t_2, A_0, a_2, q_1 \rangle$,

$A_2 = O(A_2) \langle r_1, d_1, t_1, A_0, a_1, q_2 \rangle$, $K_1 = \langle d_2, d_{2+}, t_2, a_2, q_1 \rangle$,

$K_2 = \langle d_1, d_{1+}, t_1, a_1, q_2 \rangle$.

Finally we put $T_1 = R_1 R_2 \langle t_1, t_2 \rangle$, $K_{12} = \langle d_2, d_{2+}, t_2 \rangle$, $L_3 = \langle a_1, a_2 \rangle$, $S_3 = \langle a_1, a_3 \rangle$, $A = \langle S_3, b_1 t_1, b_2 t_1, y \rangle$, and $B = \langle S_3, b_1, b_2, t_1 \rangle$; thus, $A/O(A) \cong \mathrm{Alt}_7$, $B/O(B) \cong \Sigma_6$, and $B' = O(B') \langle S_3, b_1 t_1, b_2 t_1 \rangle$.

1. The 2-Local Subgroup $N_G(V)$

By results of Part I we immediately get the following.

(1.1) (a) Without loss of generality we may assume that there are elements v_1, v_2 in V such that $(L! \, V) = 155 v_1 + 868 v_2$, $C_L(v_1) = O(L) \, N_L(R_1)$, and $C_L(v_2) = O(L) \, (E_2 \cdot B)$; moreover $Z(VT) = \langle v_1 \rangle$.

(b) If $f \in L - O(L)$ with $f^5 \in O(L)$, then $C_V(f)^{\#} \subseteq \mathrm{ccl}_L(v_2)$ and $| C_V(f)| = 4$.

(c) If $q \in L - O(L)$ with $q^7 \in O(L)$, then $C_V(q)^{\#} \subseteq \mathrm{ccl}_L(v_1)$ and $| C_V(q)| = 2$.

(d) If $d \in L - O(L)$ with $d^{31} \in O(L)$, then $C_V(d) = 1$.

(1.2) We have $| C_V(x)| = 2^4$ for any $x \in L - O(L)$ with $x^3 \in O(L)$, $C_V(S_3) = \langle v_1, v_2 \rangle$, and $C_V(L_3) = \langle v_1, v_1^{t_0} \rangle$. Moreover $v_1 v_2 = v_1^{b_1}$ and $v_1 v_1^{t_0} \in \mathrm{ccl}_L(v_2)$.

Proof. Without loss of generality we may assume that $O(L) = 1$. Let x denote an element of L with $o(x) = 3$; thus, $c = | C_V(x)| = 2^{2n}$ with $0 \leqslant n \leqslant 4$.

If $x \in \mathrm{ccl}_L(a_1)$, then $c = 2^4$, because $\langle v_1 \rangle = C_V(a_1, q_1)$ and $\langle a_1, q_1 \rangle \cong Z_{21}$. Now suppose that $x \in \mathrm{ccl}_L(a_1 a_3)$. Since $C_L(a_1 a_3) \cong Z_3 \times \mathrm{Alt}_5$, $v_1 \in C_V(a_1 a_3)$, and $Z_{15} \not\leqslant C_L(v_2)$, we get $c \in \{2^4, 2^8\}$. The fact that $x_1 = r_2 d_{1+} a_3 q_0{}^2 a_2{}^2 \in \mathrm{ccl}_L(a_1 a_3)$ and $o(a_1 x_1{}^2 a_1 x_1) = 7$ now yields $c = 2^4$.

Clearly, $N_L(S_3) = S_3 \cdot \langle t_1, d_{2+}, b_1 \rangle$ controls the L-fusion of the elements in $C_V(S_3)$. As $\langle v_1, v_2 \rangle \leqslant C_V(S_3)$ and $N_L(S_3) \leqslant C_L(v_2)$ as well as $N_L(S_3) \cap C_L(v_1) = S_3 \cdot \langle t_1, d_{2+} \rangle$, we get $C_V(S_3) = \langle v_1, v_2 \rangle$ and $v_1^{b_1} = v_1 v_2$. Note that $t_0 \in N_L(L_3) - C_L(v_1)$; hence, $C_V(L_3) = \langle v_1, v_1^{t_0} \rangle$ and $v_1 v_1^{t_0} \in \mathrm{ccl}_L(v_2)$. Q.E.D.

(1.3) The group $C_L(v_1)$ acts on V as follows:

$$(C_L(v_1)! \, V) = 1 v_1 + 42 v_1^{q_0} + 112 v_1 v_2 + 84 v_2{}^y + 112 v_2 + 672 v_2{}^d.$$

Proof. Without loss of generality we may assume that $O(L) = 1$. So $L_1 = C_L(v_1) = N_L(R_1)$.

Calculating $(L_1)^{q_0}$ we easily see that $C_L(v_1, v_1^{q_0}) =$

$$\left\{ X = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 \\ * & * & * & * & * \\ * & 0 & 0 & 1 & 0 \\ * & * & * & * & * \end{pmatrix} \middle| \, * \in GF(2), \det(X) = 1 \right\}.$$

Thus, $|C_L(v_1, v_1^{q_0})| = 2^9 \cdot 3$ and $|\mathrm{ccl}_{L_1}(v_1^{q_0})| = 2 \cdot 3 \cdot 7 = 42$.

Let $D = B \cap L_1$; then $C_L(v_1, v_2) = E_2 \cdot D$ and $S_3 \cdot \langle t_1, d_{2+} \rangle \leqslant D$ as well as $2^2 \cdot 3^2 \mid |D| \mid 2^3 \cdot 3^2$, because $5 \nmid |D|$ and B contains no subgroups with index 5. Supposing $|D| = 2^3 \cdot 3^2$ we conclude $D = N_B(S_3) = N_L(S_3) \leqslant L_1$ and thus, by Sylow's theorems, $2^7 \cdot 3 \equiv 1 \bmod 3$, which is obviously absurd. So $D = S_3 \cdot \langle t_1, d_{2+} \rangle$ and $|\mathrm{ccl}_{L_1}(v_1 v_2)| = |\mathrm{ccl}_{L_1}(v_2)| = 2^4 \cdot 7 = 112$.

Clearly, $C = C_L(v_1, v_2{}^d) = L_1 \cap (E_2 B)^d$ and $|C| \mid 2^8 \cdot 3^2$. Since $E_2{}^d \cap L_1 = \langle r_0{}^d \rangle \lhd C$, $|C|$ divides $2^5 \cdot 3$. Now $|L_1 \colon C| \leqslant 1023$ implies $|C| = 2^5 \cdot 3$ and $|\mathrm{ccl}_{L_1}(v_2{}^d)| = 2^5 \cdot 3 \cdot 7 = 672$.

Now let $D = B^y \cap L_1$; then $C_L(v_1, v_2{}^y) = L_1 \cap E_2 B^y = E_2 \cdot D$. Some easy calculations yield y: $a_1 a_3 \to a_1 a_3$, $t_1 d_{2+} \to t_1 d_{2+}$, $b_1 \to r_3 d_1$, $b_2 \to r_1 t_1 d_{2+}$, and $a_1 a_3 \cdot r_3 d_1 \cdot r_1 t_1 d_{2+} \cdot a_1 a_3 = r_1 t_1 d_{2+} \cdot d_{1+}$. Hence, $S = \langle r_1 d_{1+}, r_1 r_3 d_1, t_1 d_{2+} \rangle \in \mathrm{Syl}_2((B^y)')$ and $S \times \langle r_1 \rangle \in \mathrm{Syl}_2(B^y) \cap \mathrm{Syl}_2(D)$. Since D contains no subgroups with index 5 and $a_1 a_3 \in D$, we finally get $|D| = 2^4 \cdot 3$ and $|\mathrm{ccl}_{L_1}(v_2{}^y)| = |L_1 \colon E_2 D| = 2^2 \cdot 3 \cdot 7 = 84$. Q.E.D.

(1.4) Let $Q = O(L)$, $\bar{L} = L/Q$, and $L_1 = C_L(v_1)$. Then the following statements hold.

(a) $\bar{L}_1 = N_{\bar{L}}(\bar{R}_1)$ contains exactly three conjugacy classes of elements of order three with representatives $\bar{a}_1$, $\bar{a}_3$, and $\bar{a}_1 \bar{a}_3$.

(b) $(C_{\bar{L}}(\bar{a}_1)! \, C_V(a_1)) = (C_{\bar{L}_1}(\bar{a}_1)! \, C_V(a_1)) = 1v_1 + 7v_1 v_2 + 7v_2$.

(c) $(C_{\bar{L}}(\bar{a}_3)! \, C_V(a_3)) = 1v_1 v_2 + 7v_1 + 7v_2$ and $(C_{\bar{L}_1}(\bar{a}_3)! \, C_V(a_3)) = 1v_1 v_2 + 1v_1 + 6v_1^* + 1v_2 + 6v_2^*$ with $v_1^* \in \mathrm{ccl}_{L_1}(v_1^{q_0})$ and $v_2^* \in \mathrm{ccl}_{L_1}(v_2{}^d)$.

(d) $(C_{\bar{L}}(\bar{a}_1 \bar{a}_3)! \, C_V(a_1 a_3)) = 5v_1 + 10v_2$ and $(C_{\bar{L}_1}(\bar{a}_1 \bar{a}_3)! \, C_V(a_1 a_3)) = 1v_1 + 4v_1 v_2 + 4v_2 + 6v_2{}^y$.

Proof. Without loss of generality we may assume that $Q = 1$. So $L_1 = N_L(R_1) = R_1 \cdot (K_1 \times \langle a_1, t_1 \rangle) \cong E(2^6) \cdot (GL_3(2) \times \Sigma_3)$.

As $C_L(a_1) = \langle a_1 \rangle \times K_1 < L_1$ and $C_L(a_1, v_2) = \langle a_1 \rangle \times \langle d_2, t_2, a_3, d_{2+} \rangle \cong Z_3 \times \Sigma_4$, the conclusions of (a) and (b) hold.

Now calculate $C_L(a_1 a_3, v_1) = \langle a_1 a_3 \rangle \times \langle r_3 d_1, r_1 d_1 d_{1+}, a_1 \rangle \cong Z_3 \times \mathrm{Alt}_4$, $C_L(a_1 a_3, v_2) = S_3 \langle b_1 \rangle$, and $C_L(a_1 a_3, v_1, v_2) = S_3$; since $\langle v_1, v_2, v_2{}^y \rangle \leqslant C_V(a_1 a_3)$ and $C_L(a_1 a_3, v_1, v_2{}^y) = \langle a_1 a_3, r_3 d_1 \rangle$, the statements of (d) follow.

Let $D = C_L(a_3)$. Clearly, $\langle v_1, v_2 \rangle \leqslant C_V(a_3)$. Now calculate $C_L(a_3, v_1) =$

$C_L(a_3, v_2) = \langle a_3\rangle \times \langle r_0, r_2, a_1, t_1\rangle \simeq Z_3 \times \Sigma_4$. Since $a_3 \in \mathrm{ccl}_L(a_1)$, we easily conclude that $(D!\, C_V(a_3)) = 1w + 7v_1 + 7v_2$ with some $w \in \mathrm{ccl}_L(v_1)$.

As $k = t_0 r_3 t_0 \in C_L(a_3) - L_1$, we calculate $C_L(v_1, v_1{}^k) =$

$$\left\{ X = \begin{pmatrix} 1 & * & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & 0 & 0 & 1 \end{pmatrix} \mid * \in GF(2), \det(X) = 1 \right\}; \quad \text{so} \quad v_1{}^k \in \mathrm{ccl}_{L_1}(v_1^{q_0}).$$

Since $C_L(a_3, v_1, v_1{}^k) = \langle a_3\rangle \times \langle r_2, t_1 a_1\rangle \simeq Z_3 \times E(2^2)$, we get $(C_L(a_3, v_1)!\, \mathrm{ccl}_D(v_1)) = 1v_1 + 6v_1{}^k$ and $w = v_1 v_2$.

Obviously $v_2 \neq v_2{}^k \in C_V(a_3)$. Since $C_L(v_1, v_2{}^k) = L_1 \cap E_2{}^k \cdot B^k \trianglerighteq L_1 \cap E_2{}^k = \langle r_2{}^k\rangle = \langle r_2 t_1 a_1\rangle$, application of (1.3) yields $v_2{}^k \in \mathrm{ccl}_{L_1}(v_2{}^d)$. Moreover, $C_L(a_3, v_1, v_2{}^k) \leqslant C_L(a_3, v_1, r_2{}^k) = \langle a_3\rangle \times \langle r_2, t_1 a_1\rangle$. Hence, the results claimed in (c) hold. Q.E.D.

(1.5) The following statements hold:

(a) If $x \in \{E_1, R_1, R_2\}$, then $C_V(X)^{\#} \subseteq \mathrm{ccl}_L(v_1)$.

(b) $C_V(R_1) = \langle v_1\rangle$, $C_V(E_1E_2) = \{1, v_1, v_1^{q_0}, \ldots, v_1^{q_0^6}\}$, $C_V(E_1) = C_V(E_1E_2) \cdot \langle v_1^{q_0 q_1}\rangle$, and $C_V(E_2) \simeq E(2^6)$.

(c) $C_V(R_2) = \{1, v_1, v_1^{q_2}, \ldots, v_1^{q_2^6}\} < C_V(E_2)$ and $C_V(R_2, E_1) = C_V(R_2, E_1E_2) = \{1, v_1, v_1^{q_0^2}, v_1^{q_0^6}\}$ with $v_1^{q_0^2} = v_1^{q_2^2}$ and $v_1^{q_0^6} = v_1^{q_2^6}$.

(d) Let $L_1 = C_L(v_1)$; then

$$(N_L(E_2)!\, C_V(E_2)) = 35v_1 + 28v_2,$$

$$(N_{L_1}(E_2)!\, C_V(E_2)) = 1v_1 + 18v_1^{q_0^2} + 16v_1v_2 + 16v_2 + 12v_2{}^y,$$

$$(N_{L_1}(E_1)!\, C_V(E_1)) = 1v_1 + 14v_1^{q_0^2},$$

$$(N_{L_1}(R_2)!\, C_V(R_2)) = 1v_1 + 6v_1^{q_0^2},$$

and

$$(N_{L_1}(E_1E_2)!\, C_V(E_1E_2)) = (H_0 \cap L!\, C_V(E_1E_2)) = 1v_1 + 6v_1^{q_0^2}.$$

Proof. Again we may assume that $O(L) = 1$. Thus, $L_1 = C_L(v_1) = N_L(R_1)$. Since $E_1 \notin \mathrm{ccl}_L(E_2)$ and $C_T(v_2{}^y) = E_2 \cdot \langle r_1, r_3 d_1, d_{1+}, t_1 d_{2+}\rangle$ is an S_2-subgroup of $C_L(v_2{}^y)$, the statements in (a) hold.

As $v_1 \in C_V(R_1) \lhd V \cdot L_1$ and $C_V(R_1)^{\#} \subseteq \mathrm{ccl}_L(v_1)$, application of (1.3) yields $C_V(R_1) = \langle v_1\rangle$.

Let $i \in \{1, 2\}$ and $c_i = |\, C_V(E_i)|$. Since $v_1 \in C_V(E_1)$ and $q_1 \in N_L(E_1) \cap L_1$, we get $c_1 \equiv 2 \bmod 7$ and $c_1 \equiv 0 \bmod 5$; this implies $c_1 = 16$. We know that

$\langle v_1, v_2, v_2^y\rangle \leqslant C_V(E_2)$. If $c_2 = 2^9$, we get $C_V(E_2) = C_V(r_0) \cong E(2^9)$ and thus $2^6 \leqslant c_1$, a contradiction. So $c_2 = 2^6$, because A_2 acts on $C_V(E_2)$.

We have $q_0 \in H_0 = N_L(E_1) \cap N_L(E_2)$, $q_0 \notin L_1$, and $q_0q_1\{1, q_0, \ldots, q_0^6\} \cap L_1 = \varnothing$. This finally proves (b).

Now let $W = C_V(R_2)$ and $w = |W|$. Since $W^\# \subseteq \mathrm{ccl}_L(v_1)$ and $q_2 \in N_L(R_2) - L_1$, we get $E(2^3) \leqslant \langle v_{12}^{q_2^i} \mid 0 \leqslant i \leqslant 6\rangle \leqslant W$ and $w \in \{2^3, 2^4, 2^6, 2^7\}$. Moreover, $L_3 \in \mathrm{Syl}_3(N_{VL}(R_2))$ and $C_W(L_3) = \langle v_1\rangle$. As $N_{VL}(R_2) \cap N_{VL}(L_3) = C_V(R_2, L_3) \cdot (N_L(R_2) \cap N_L(L_3))$, applications of Sylow's theorems yield $|W: \langle v_1\rangle| \equiv 1 \bmod 3$ and thus $w \equiv 2 \bmod 3$. This implies $w = 2^3$ or $w = 2^7$. If $w = 2^7$, then $E(2^4) \leqslant W \cap W^k \leqslant C_V(r_0r_0^k)$ for any $k \in L$; putting $k = t_0r_3t_0$ we derive a contradiction, because $o(r_0r_0^k) = 3$ and $(W \cap W^k)^\# \subset \mathrm{ccl}_L(v_1)$. Therefore $w = 2^3$.

Trivial calculations now yield the final conclusions of (c).

The following statements can be easily verified:

(1) $C_L(v_2) = E_2 \cdot B < N_L(E_2) = E_2 \cdot A_2$, $L_1 \cap E_2A_2 = R_1E_2 \cdot S_3 \cdot \langle t_1, d_{2+}\rangle$, $C_L(v_1, v_2) \cap E_2A_2 = C_L(v_1, v_2)$, $C_L(v_1, v_2^y) \cap E_2A_2 = C_L(v_1, v_2^y)$, and $C_L(v_1, v_{10}^{q_0^2}) \cap E_2A_2 = E_1E_2R_2$;

(2) $E_1A_1 \cap L_1 = R_1 \cdot (K_1 \times \langle t_1\rangle)$ and $|C_L(v_1, v_{10}^{q_0^2})| = 2^9 \cdot 3$;

(3) $N_L(R_1) \cap N_L(R_2) = R_1R_2 \cdot L_3 \cdot \langle t_1, t_2\rangle$;

(4) $L_1 \cap H_0 = L_1 \cap N_L(E_1) \cap N_L(E_2) = E_1E_2R_1\langle a_3, d_{2+}\rangle$ and $C_{H_0}(v_1, v_{10}^{q_0^2}) = E_1E_2R_2$.

From that we deduce the results claimed in (d). Q.E.D.

Notation. For convenience we use the following designations throughout the rest of this paper:

$k_1 = q_1d_{2+}$, $k_2 = t_0r_3t_0$, $k_3 = t_2a_2$, $k_4 = a_1^2q_1^4$, $k_5 = a_1^2d_{2+}d_2a_3$, and $k_6 = q_1^3a_2$.

Moreover, let $V_0 = \langle \mathrm{ccl}_{C_L(v_1)}(v_2^y)\rangle$, $V_1 = [C_V(a_1), q_1]$, and $E = V_0 \cdot R_1$.

(1.6) Clearly, $L_1 = C_L(v_1)$ contains T. Table II presents the T-orbits in $V - \langle v_1\rangle$.

Outline in Proof. Without loss of generality we may assume that $O(L) = 1$. If $x \in L$ and $i \in \{1, 2\}$, then determine $C_T(v_i^x)$ by calculating $T^{x^{-1}} \cap C(v_i)$. As these operations are trivial, we do not state them here.

(1.7) Let $L_1 = C_L(v_1)$; then the following hold:

(a) $V_0 = \langle \mathrm{ccl}_{L_1}(v_1^{q_0})\rangle = \langle C_V(X) \mid X \in \mathrm{ccl}_{L_1}(E_1)\rangle \cong E(2^7)$ and $V_0 \lhd V \cdot L_1$.

(b) $E \cong Ex^+(2^{13})$ and $N_{VL}(E) = V \cdot N_L(R_1)$.

(c) $V_1 = [C_V(a_1), K_1] = \{1, v_1v_{21}^{q_1^i} \mid 0 \leqslant i \leqslant 6\} \cong E(2^3)$, $V_1K_1/O(K_1) \cong \mathrm{Hol}(E(2^3))$, and $V = V_0 \times V_1$. Moreover, we have $N_{VL}(E) = O(N_L(R_1))EV_1K_1\langle a_1, t_1\rangle$ and $C_{VL}(v_1) = O(L)\, N_{VL}(E)$.

TABLE II

T-Orbits in $V - \langle v_1 \rangle$

	Representative x of a T-orbit	$\lvert \mathrm{ccl}_T(x) \rvert$	$C_T(x)$
$\mathrm{ccl}_{L_1}(v_1^{q_0})$	$v_1^{q_0{}^2}$	2	$E_1E_2R_2$
	$v_1^{q_0}$	4	$E_1E_2\langle d_{1+}\rangle$
	$v_1^{q_2}$	4	$R_2E_2\langle d_{1+}\rangle$
	$v_1^{q_0q_1}$	8	$E_1\langle r_3, d_{1+}, d_{2+}\rangle$
	$v_1^{q_0q_2}$	8	$E_2\langle r_3, r_1d_1, d_{1+}\rangle$
	$v_1^{k_2}$	16	$\langle r_1, r_2, r_3, d_1, d_{1+}, d_{2+}\rangle$
$\mathrm{ccl}_{L_1}(v_2{}^y)$	$v_2{}^y$	4	$E_2\langle r_1, r_3d_1, d_{1+}, t_1d_{2+}\rangle$
	$v_2^{ya_1}$	8	$E_2\langle r_1r_3, d_1, r_1d_{1+}\rangle$
	$v_2^{yk_3}$	8	$\langle r_0, r_1, r_2r_3, r_3d_1, d_{1+}, t_1d_{2+}, t_1d_2\rangle$
	$v_2^{yq_1}$	16	$\langle r_0, r_1r_2, r_3, r_1d_1, d_{1+}, d_2t_1t_2\rangle$
	$v_2^{yq_1{}^2}$	16	$\langle r_0r_1, r_2, r_3, r_0d_1, r_0d_{1+}, d_2d_{2+}\rangle$
	$v_2^{yq_1{}^3}$	32	$\langle r_0r_1, r_2, r_0r_3, d_1, r_0d_{1+}\rangle$
$\mathrm{ccl}_{L_1}(v_1{}^iv_2)$, $i \in \{1, 2\}$	$v_1{}^iv_2$	16	$E_2\langle t_1, d_{2+}\rangle$
	$v_1{}^iv_2^{k_1{}^2}$	32	$\langle r_1, r_3, d_{2+}, d_2, t_1\rangle$
	$v_1^{i+1}v_1^{t_0}$	64	$\langle d_1, d_{1+}, t_1, t_2\rangle$
$\mathrm{ccl}_{L_1}(v_2{}^d)$	$v_2^{dk_4}$	32	$\langle r_0, t_1\rangle \times \langle d_{2+}, r_1d_1d_2t_2\rangle$
	$v_2^{dk_5}$	64	$\langle r_1, t_1, r_0d_{2+}, d_2\rangle$
	$v_2^{dk_6}$	64	$\langle r_0r_2\rangle \times \langle d_{2+}, d_1d_{1+}t_2\rangle$
	$v_2^{dq_1{}^3}$	128	$\langle r_1r_3, r_0r_2d_2, d_{2+}d_2\rangle$
	$v_2^{da_1{}^2}$	128	$\langle r_0r_1d_1, t_1, d_2t_2\rangle$
	$v_2{}^d$	256	$\langle r_0r_1r_2r_3d_1d_{1+}, d_2t_2\rangle$

Proof. Put $C_i = C_V(E_1)^{a_1^i}$ and $D_i = \langle v_1, v_{10}^{q_0^2}\rangle^{a_1^i}$ for $i \in \{0, 1, 2\}$. Then $D_i = C_i \cap C_V(R_2)$, because $a_1 \in N_L(R_2)$.

Let $i, j \in \{0, 1, 2\}$ with $i \neq j$. If $C_i = C_j$, then $C_0 = C_1 = C_2$ and therefore $C_0 \leqslant C_V(E_1, E_1^{a_1}) \leqslant C_V(R_1) = \langle v_1 \rangle$ which is impossible. Thus, $C_i \cap C_j = D_i \cap D_j = \langle v_1 \rangle$ and $D_iD_j = C_V(R_2)$.

Clearly, K_1 acts on $C_0 \cap C_1C_2$. As $D_0 \leqslant C_0 \cap C_1C_2$ and $\lvert C_0 \cap C_1C_2 \rvert \equiv \lvert C_0C_1C_2 \rvert \equiv 2 \bmod 7$, we get $C_0 \leqslant C_1C_2$ and so $C_0C_1C_2 = C_iC_j \cong E(2^7)$. Furthermore, $\mathrm{ccl}_{L_1}(C_V(E_1)) = \{C_0, C_1, C_2\}$, because $L_1 \cap O(L)\, N_L(E_1) = O(L)\, R_1E_1K_1$. Hence, $C_0C_1C_2$ is a normal subgroup of $V \cdot L_1$ contained in V. By (1.3), $C_0C_1C_2 = V_0$ and $(L_1 ! V_0) = 1v_1 + 42v_1^{q_0} + 84v_1{}^y$. This proves (a).

Obviously, $\lvert E \rvert = 2^{13}$ and, by (1.6), $C_{R_1}(V_0) = 1$. Thus, $Z(E) = C_{V_0}(R_1) =$

$\langle v_1\rangle$. Moreover, $V_0 > \phi(E) = \langle x^2 \mid x \in E\rangle = [V_0, R_1] = E' \geqslant \langle v_1\rangle$. As $N_L(R_1)$ acts on $[V_0, R_1]$ and $(N_L(R_1)!\ V_0) = (L_1!\ V_0) = 1v_1 + 42v_1^{q_0} + 84v_2{}^y$, we get $\phi(E) = E' = Z(E) = \langle v_1\rangle$. Therefore $E \cong Ex^+(2^{13})$.

Now let $X = [V, R_1]$; clearly, $V > \phi(VR_1) = (VR_1)' = X \geqslant \langle v_1\rangle$. Recall the statement of (1.3); since L_1 normalizes X, we get $X \in \{\langle v_1\rangle, V_0\}$. In particular, E is a normal subgroup of VT. Note that $O(L)\,N_L(R_1)$ is a maximal subgroup of L; hence, we easily see that $N_{VL}(E) = VN_L(R_1)$.

Put $C = C_V(a_1)$. Since $N_L(\langle a_1\rangle \bmod O(L)) = O(L)\,K_1\,\langle a_1, t_1\rangle$ and $[K_1, \langle a_1, t_1\rangle] \leqslant O(L)$, we have $C \leqslant C_V(t_1)$ and $C = \langle v_1\rangle \times V_1$ with $V_1 = [C, q_1] \cong E(2^3)$. By (1.4), $C \cap V_0 = \langle v_1\rangle$ and $V_1{}^{\#} = \mathrm{ccl}_{K_1}(v)$ with a suitable element v of $v_2\langle v_1\rangle$. In particular, we get $V = V_0 \times V_1$ and $N_{VL}(E) = O(N_L(R_1))\,EV_1K_1\langle a_1, t_1\rangle$ with $[V_1, \langle a_1, t_1\rangle] = 1$ and $V_1K_1/O(K_1) \cong \mathrm{Hol}(E(2^3))$.

As $v_1 \in [V, r_0]$ and $q_0 \in H_0 - L_1$, $C_V(r_0) = C_V(E_1)\,C_V(E_2) \cong E(2^7)$ and $[V, r_0] = C_V(E_1E_2) \cong E(2^3)$. Moreover, we have $C_{K_1}(V_1) = O(K_1)$ and $t_2 \in \mathrm{ccl}_L(r_0)$; therefore $\varnothing \neq [V, t_2]^{\#} \subset \mathrm{ccl}_L(v_1)$. This implies $V_1{}^{\#} = \mathrm{ccl}_{K_1}(v_1v_2) = \mathrm{ccl}_{\langle q_1\rangle}(v_1v_2)$. Q.E.D.

(1.8) (a) Writing $(T; X)$ instead of $(T!\ C_V(X))$ for the sake of convenience we get the following:

$$(T; E_1E_2) = 1v_1 + 2v_1^{q_0^2} + 4v_1^{q_0},$$

$$(T; R_2) = 1v_1 + 2v_1^{q_0^2} + 4v_1^{q_2},$$

$$(T; \langle r_0, r_1, r_2\rangle) = 1v_1 + 2v_1^{q_0^2} + 4v_1^{q_0} + 4v_1^{q_2} + 4v_2{}^y,$$

$$T; \langle r_0, r_1\rangle) = C_V(r_0, r_1, r_2) + 8v_1^{q_0q_1} + 8v_2^{yk_3},$$

$$(T; \langle r_0, r_2\rangle) = C_V(r_0, r_1, r_2) + 8v_1^{q_0q_2} + 8v_2^{ya_1} + 16v_1v_2 + 16v_2,$$

$$(T; \langle r_0, r_1r_2\rangle) = C_V(r_0, r_1, r_2) + 16v_2^{ya_1},$$

$$(T; \langle r_0\rangle) = C_V(r_0, r_2) + 8v_1^{q_0a_1} + 8v_2^{yk_3} + 16v_2^{ya_1} + 32v_2^{dk_4};$$

moreover, $C_V(R_1 \cap R_2) = C_V(R_2)$ and $C_V(r_0, r_2) = C_V(E_2)$.

(b) V_0 contains $U = C_V(r_1r_2)$ with index two and $C_U(a_1a_2) = \langle v_1, v_2^{yk_1}\rangle$; moreover, we have

$$(H_1!\ U) = 1v_1 + 6v_1^{q_2} + 12v_1^{q_0} + 8v_2^{ya_1} + 12v_2{}^y + 24v_2^{ya_1{}^3t_1}$$

and

$$(T_1!\ U) = 1v_1 + 2v_1^{q_0^2} + 4v_1^{q_2} + 4v_1^{q_0} + 8v_1^{k_2} + 8v_2^{ya_1}$$
$$+ 4v_2{}^y + 8v_2^{ya_1t_1} + 8v_2^{ya_1{}^2t_1} + 16v_2^{ya_1{}^3t_1}.$$

(c) We have $[V, R_1] = V_0$, $[V, r_0] = C_V(E_1E_2)$, and $[V, r_1r_2] = C_V(R_2) \cup \mathrm{ccl}_{T_1}(v_2^{yq_1}) \cong E(2^4)$.

Proof. It is a consequence of (1.6) that the assertions in (a) hold. Furthermore, the statements of (1.6) imply that $V_0 > C_V(r_1r_2) \cong E(2^6)$ and that T_1 acts on $C_V(r_1r_2)$ as stated above, because $\mathrm{ccl}_T(r_1r_2) = r_1r_2\langle r_0\rangle$ and $C_T(r_1r_2) = T_1$.

Let $U = C_V(r_1r_2)$, $U_0 = C_U(a_1a_2)$, $X = U \cdot H_1$, and $\bar{X} = X/O(X)$; clearly, $O(X) = O(H_1) \leqslant O(L)$.

Since $k_1 = q_1d_{2+}$ and $(a_1a_3)^{k_1} \in a_1a_2 \cdot O(L)$, we get $\langle v_1, v_2^{yk_1}\rangle \leqslant U_0 \leqslant C_{V_0}(a_1a_2) \cong E(2^3)$ by (1.4). As $N_{\bar{X}}(\langle \bar{a}_1\bar{a}_2\rangle) = \bar{U}_0N_{\bar{H}_1}(\langle \bar{a}_1\bar{a}_2\rangle)$, applications of Sylow's theorems yield $| U: U_0 | \equiv 1 \bmod 3$. Hence, $U_0 = \langle v_1, v_2^{yk_1}\rangle$.

We know that $U_0 \cap \mathrm{ccl}_L(v_1) = \{v_1\}$ and $U_0 \cap \mathrm{ccl}_L(v_2) = v_2^{yk_1}\langle v_1\rangle$; thus, $\mathrm{ccl}_{H_1}(v_1^{q_2}) = \mathrm{ccl}_{T_1}(v_1^{q_0^2}) \cup \mathrm{ccl}_{T_1}(v_1^{q_2})$ and $\mathrm{ccl}_{H_1}(v_1^{q_0}) = \mathrm{ccl}_{T_1}(v_1^{q_0}) \cup \mathrm{ccl}_{T_1}(v_1^{k_2})$, because a_1a_2 acts on $C_V(R_2)$. As $H_1 = O(H_1)\langle T_1, a_1a_2\rangle$, $\mathrm{ccl}_{H_1}(v_2^{yq_1}) = \mathrm{ccl}_{T_1}(v_2^{yq_1})$. Clearly, $C_{H_1}(v)/O(H_1)$ is a 2-group for each v of $\{v_2{}^y, v_2^{yq_1t_1}, v_2^{yq_1{}^2t_1}, v_2^{yq_1{}^3t_1}\}$. Furthermore, $O(H_1) \cdot R_2$ is a normal subgroup of H_1 and $| C_{R_2}(v_2{}^y)| = | C_{R_2}(v_2^{yq_1{}^2t_1})| = 2^4 \neq 2^3 = | C_{R_2}(v_2^{yq_1{}^3t_1})| = | C_{R_2}(v_2^{yq_1t_1})|$. Hence, the H_1-fusion of the elements in U has been solved.

We have already shown that $[V, r_0] = C_V(E_1E_2)$ and $[V, R_1] \in \{\langle v_1\rangle, V_0\}$; therefore $[V, R_1] = V_0$.

As $K = [V, r_1r_2] \cong E(2^4)$ and $v_1 \in K \leqslant U$, application of a theorem of Maschke yields $U_0 \leqslant K$. Since H_1 acts on K, we get $(H_1! K) = 1v_1 + 6v_1^{q_2} + 8v_2^{yq_1}$. So $K = C_V(R_2) \cup \mathrm{ccl}_{T_1}(v_2^{yq_1})$. Q.E.D.

(1.9) The following assertions hold:

$$[V, r_2] = \langle v_1, v_1^{q_2}, v_1^{q_0q_2}\rangle, \qquad [V, r_0r_2] = \langle v_1, v_1^{q_2^4}, v_1^{q_0q_2t_1}\rangle,$$

$$[V, d_2] = \langle v_1^{q_0^2}, v_1^{q_2}, v_1v_2\rangle, \qquad [V, t_2] = \langle v_1v_1^{q_0}, v_1^{q_0q_2d_{2+}}, v_1v_2\rangle,$$

$$[V, r_1] = \langle v_1, v_1^{q_0^2}, v_1^{q_0q_1}\rangle, \qquad [V, r_3] = \langle v_1, v_1^{q_2}, v_1^{k_2}\rangle,$$

$$[V, d_1] = \langle v_1, v_1^{q_0}, v_1^{q_0q_1}\rangle, \qquad [V, d_{1+}] = \langle v_1, v_1^{q_0q_2d_{2+}}, v_1^{k_2}\rangle,$$

$$[V, t_1] = \langle v_1^{q_0^2}, v_1v_1^{q_0}, v_1v_1^{q_0q_1}\rangle, \qquad [V, d_{2+}] = \langle v_1^{q_0^2}, v_1^{q_2}, v_1v_2^{k_1^2}\rangle.$$

Proof. We know that $[V, r_0] = C_V(E_1E_2)$ and $a_1 \in N_L(R_1) \cap N_L(R_2)$ with $a_1\colon v_1 \to v_1$, $v_1^{q_2} \to v_1^{q_0^2} \to v_1^{q_2^4}$, $r_0 \to r_0r_2 \to r_2$.

Since $q_0a_1{}^2r_1d_{2+}q_2{}^6q_0{}^6, q_0a_1r_1d_{2+}t_1q_2{}^6q_0{}^6 \in O(L)\, N_L(R_1) = C_L(v_1)$ and $[\langle r_0, r_2\rangle, \langle r_1, d_{2+}\rangle] = 1$, the first two statements hold.

As $q_2\colon r_0 \to r_2 \to d_2$, $[r_1, d_2] = 1$, and $q_0q_2{}^2r_1b_1 \in C_L(v_1)$, the group $[V, d_2]$ contains the following elements: $v_1^{q_2^2} = v_1^{q_0^2}$, $v_1^{q_0^2q_2^2} = v_1^{q_2^4} = v_1^{q_0^2}v_1^{q_2}$, and $v_1^{q_0q_2^2r_1} = v_1^{b_1} = v_1v_2$.

We have $d_2^{q_0} = t_2$, $q_0 \in b_2a_3{}^2d_1d_{1+} \cdot O(L)$, $[t_2, \langle d_1, d_{1+}\rangle] = 1$, $(v_1v_2)^{q_0} = v_1^{q_0} \cdot v_2^{d_1d_{1+}}$, and $v_1^{q_2q_0} = v_1^{q_0q_1d_{2+}}$; hence, $[V, t_2] = \langle v_1v_1^{q_0}, v_2^{q_0q_2d_{2+}}, v_1v_2\rangle$.

The action of $\langle a_1, q_1\rangle$ on R_1 is well known and $q_1\colon v_1 \to v_1$, $v_1^{q_0^2} \to v_1 v_1^{q_0} v_1^{q_0^2}$, $v_1^{q_0 q_1} \to v_1 v_1^{q_0^2}$; keeping this in mind it is easy to calculate $[V, x]$ for $x \in \{r_1, d_1, r_3, d_{1+}\}$.

We know the action of q_0 on E_1 and the action of q_2 on R_2; furthermore, $[d_2, t_1] = [r_0, d_{2+}] = 1$. As $v_1^{q_0 q_1 q_0^2} = v_1 v_1^{q_0 q_1 d_2}$ and $v_1^{q_0 q_1 q_2^2} = v_1^{b_1 k_1^2 r_0} = v_1 v_2^{k_1^2 r_0}$, we easily verify the last two statements. Q.E.D.

(1.10) Investigating the fusion of 2-elements it will be very useful to know the following:

(a) $\mathrm{ccl}_T(v_1^{q_0^2}) = v_1^{q_0^2}\langle v_1\rangle$, $\mathrm{ccl}_T(v) = v\langle v_1, v_1^{q_0^2}\rangle$ for each v of $\{v_1^{q_0}, v_1^{q_2}, v_2^{y}\}$, $\mathrm{ccl}_T(v_2^{ya_1}) = v_2^{ya_1} \cdot C_V(r_0, r_1, r_2)$, and $v_2^{y} = v_1^{q_0} \cdot v_1^{q_2}$.

(b) We have $v_2^{yk_3} = v_1 v_2^{y} v_1^{q_0 q_1}$ and $\mathrm{ccl}_T(v) = v \cdot C_V(E_1E_2)$ for $v \in \{v_1^{q_0 q_1}, v_2^{yk_3}\}$.

(c) $\mathrm{ccl}_T(v_1^{q_0 q_2}) = v_1^{q_0 q_2} \cdot \{\langle v_1, v_1^{q_0} v_1^{q_0^2}\rangle \cup v_1^{q_2}\langle v_1, v_1^{q_0}\rangle\}$, $v_1^{q_0^2} v_1^{q_0 q_2} \in \mathrm{ccl}_T(v_2^{ya_1}) = v_1^{q_0^2} \cdot \mathrm{ccl}_T(v_1^{q_0 q_2})$, and $v_2^{ya_1} \in v_1^{q_0^2} v_2^{y} v_1^{q_0 q_2}\langle v_1\rangle$. Furthermore, $v_2^{ya_1 t_1} = v_1 v_1^{q_0} v_2^{ya_1}$, $v_2^{ya_1 d_{2+}} = v_1^{q_0^2} v_1^{q_2} v_2^{ya_1}$, and $\mathrm{ccl}_T(v_2^{ya_1}) = v_2^{ya_1}\{\langle v_1\rangle\{1, v_1 v_1^{q_0}, v_1^{q_0^2} v_1^{q_2}, v_2^{y}\}\}$.

(d) Let $W_0 = \langle v_1, v_1^{q_2}, v_1^{q_0 q_2}\rangle$ and $W_1 = \langle v_1, v_1^{q_0^2} v_1^{q_2}, v_2^{y} v_1^{q_0 q_2}\rangle$. Then $\mathrm{ccl}_T(v_1^{k_2}) = v_1^{k_2} \cdot \{W_0 \cup v_1^{q_0 q_1} \cdot W_1\}$, $v_2^{yq_1^2} \in v_1^{q_0^2} v_1^{q_0 q_2} v_1^{k_2}\langle v_1\rangle$, $v_1^{q_0^2} v_1^{k_2} \in v_2^{yq_1^2 d_2 t_2}\langle v_1\rangle$, and $\mathrm{ccl}_T(v_2^{yq_1^2}) = v_2^{yq_1^2} \cdot \{W_0 \cup v_2^{y} v_1^{q_0 q_1} \cdot W_1\} = v_1^{q_0^2} v_1^{k_2} \cdot \{W_0 \cup v_1^{q_0 q_1} \cdot W_1\}$.

(e) Let $X = \langle r_0, d_2, d_{2+}, t_2\rangle$; then $\mathrm{ccl}_T(v_2^{yq_1^3}) = \mathrm{ccl}_X(v_2^{yq_1^3}) \cup \mathrm{ccl}_X(v_2^{yq_1^3 t_1})$. We have $v_1^{q_0} v_1^{k_2} \in v_1^{q_0^2} v_1^{q_2} v_1^{q_0 q_2} v_2^{yq_1^3}\langle v_1\rangle$ and $v_1^{q_0} v_2^{yq_1^2} \in v_1^{q_2} v_2^{yq_1^3}\langle v_1\rangle$; moreover $\mathrm{ccl}_X(v_2^{yq_1^3}) = v_2^{yq_1^3}\{C_V(R_2)\langle v_1^{q_0 q_2}\rangle\}$ and $\mathrm{ccl}_X(v_2^{yq_1^3 t_1}) = v_2^{yq_1^3} v_1^{q_0 q_1} v_1^{q_0}\{C_V(R_2) \times \langle v_1^{q_0} v_1^{q_0 q_2}\rangle\}$.

(f) Let $X = \langle r_1, r_3, d_1, d_{1+}\rangle$; then $\mathrm{ccl}_T(v_1 v_2) = v_1 \cdot \mathrm{ccl}_T(v_2) = v_1 \cdot \mathrm{ccl}_X(v_2)$. Writing r^+ instead of $[v_2, r]$ for $r \in X$ we get $r_1^+ = v_1^{q_0^2}$, $r_3^+ = v_1^{q_2}$, $d_1^+ = v_1 v_1^{q_0}$, $d_{1+}^+ = v_1^{q_2} v_1^{q_0 q_2}$, $(r_1 r_3)^+ = v_1^{q_0^2} v_1^{q_2}$, $(r_1 d_1)^+ = v_1 v_1^{q_0} v_1^{q_0^2}$, $(r_1 d_{1+})^+ = v_1 v_1^{q_0^2} v_1^{q_2} v_1^{q_0 q_2}$, $(r_3 d_1)^+ = v_2^{y}$, $(r_3 d_{1+})^+ = v_1^{q_0 q_2}$, $(d_1 d_{1+})^+ = v_1 v_2^{y} v_1^{q_0 q_2}$, $(r_1 r_3 d_1)^+ = v_1^{q_0^2} v_2^{y}$, $(r_1 r_3 d_{1+})^+ = v_1 v_1^{q_0^2} v_1^{q_0 q_2}$, $(r_1 d_1 d_{1+})^+ = v_1^{q_0^2} v_2^{y} v_1^{q_0 q_2}$, $(r_3 d_1 d_{1+})^+ = v_1^{q_0} v_1^{q_0 q_2}$, and $(r_1 r_3 d_1 d_{1+})^+ = v_1 v_1^{q_0^2} v_1^{q_0} v_1^{q_0 q_2}$.

Proof. (a) Put $x = r_1 a_1 y a_1 t_1 b_2$; then $x^y \in t_1 b_2 a_1^2 b_2 \cdot O(L) \subseteq C_L(v_2)$ and $x^{q_0^6}, x^{q_2^6} \in C_L(v_1)$. Hence, $x \in C_L(v_1^{q_0}, v_1^{q_2}, v_2^{y}) - C_L(v_1)$ and so $v_2^{y} = v_1^{q_0} \cdot v_1^{q_2}$. The other assertions are consequences of (1.6) and (1.8).

(b) Consider $C_V(E_1)$ and $C_V(r_0, r_1)$. Clearly, $\mathrm{ccl}_T(v_1^{q_0 q_1}) = v_1^{q_0 q_1} \cdot C_V(E_1 E_2)$ and $v_1^{q_0 q_1} v_1^{q_2}\langle v_1^{q_0}\rangle \subseteq \mathrm{ccl}_T(v_2^{yk_3}) = \mathrm{ccl}_{\langle r_2, d_2, t_2\rangle}(v_2^{yk_3})$. Inspecting $C_T(v)$ for $v \in \mathrm{ccl}_T(v_2^{yk_3})$ we get $v_1^{q_2} v_1^{q_0 q_1} \in v_2^{yk_3 t_2}\langle v_1\rangle = v_1^{q_0} v_2^{yk_3}\langle v_1\rangle$; thus $v_2^{yk_3} \in v_2^{y} v_1^{q_0 q_1}\langle v_1\rangle$. Since $k_3 = t_2 a_2 \in (t_2)^{a_2^2} \cdot O(L)$, we get $[v_2^{ya_2}, t_2]^{a_2^2} = [v_2^{y}, k_3] = v_1^{i} v_1^{q_0 q_1} = v_1^{q_0 q_1 r_2^{i}}$ with a suitable $i \in \{0, 1\}$. Furthermore, $v_1^{q_0 q_1 r_2^{i} a_2} \in [V, t_2]^{\#} \cap C_V(E_1) = \{v_1 v_1^{q_0 q_1}\}$. Hence, $i = 1$ and $v_2^{yk_3} = v_1 v_2^{y} v_1^{q_0 q_1}$, which finally proves (b).

(c) We already know that $[V, r_0 r_2] = \langle v_1, v_1^{q_2^4}, v_1^{q_0 q_2 t_1}\rangle = \langle v_1, v_1^{q_2^4}\rangle \cup \mathrm{ccl}_{\langle r_1, d_{2+}\rangle}(v_1^{q_0 q_2 t_1})$ and $[V, r_2] = \langle v_1, v_1^{q_2}, v_1^{q_0 q_2}\rangle = \langle v_1, v_1^{q_2}\rangle \cup \mathrm{ccl}_{\langle r_1, d_{2+}\rangle}(v_1^{q_0 q_2})$.

As $[v_1^{q_0{}^2}, t_1] \in [V, t_1]^{\#} \cap C_{V_0}(E_2) \cap C_V(r_1 d_1, r_3 d_{1+}) = \{v_1 v_1^{q_0} v_1^{q_0{}^2}\} = \{v_1^{q_0{}^5}\}$, the first statement follows.

Considering $C_{V_0}(E_2)$ and $\mathrm{ccl}_T(v_1^{q_0 q_2})$ we get $v_1^{q_0{}^2} v_1^{q_0 q_2} \in \mathrm{ccl}_T(v_2^{y a_1})$. Since $C_T(v_1^{q_0{}^2} v_1^{q_0 q_2}) = E_2\langle r_3, r_1 d_1, r_1 d_{1+}\rangle$, we have $v_2^{y a_1} \in (v_1^{q_0{}^2} v_1^{q_0 q_2})^{t_1 d_{2+}}\langle v_1\rangle = v_1^{q_0{}^2} v_2{}^y v_1^{q_0 q_2}\langle v_1\rangle$.

Making use of (1.9) we now easily verify the other assertions of (c).

(d) Inspecting $[V, r_3]$ and $[V, d_{1+}]$ we find $v_1^{k_2 r_0} = v_1 v_1^{k_2}$, $v_1^{k_2 d_2} = v_1^{q_2} v_1^{k_2}$, $v_1^{k_2 t_2}, = v_1^{q_2} v_1^{q_0 q_2} v_1^{k_2}$, and thus $\mathrm{ccl}_{E_2}(v_1^{k_2}) = v_1^{k_2}\langle v_1, v_1^{q_2}, v_1^{q_0 q_2}\rangle$. Moreover, $[v_1^{k_2}, t_1] \in C_V(r_1, d_1, r_3, d_{1+}, d_{2+}) \cap [V, t_1]^{\#} = \{v_1 v_1^{q_0 q_1}\}$; making use of (1.9) and the results obtained so far in parts (a)–(c) we get $\mathrm{ccl}_{E_2}(v_1^{k_2 t_1}) = v_1^{k_2} v_1 v_1^{q_0 q_1}\langle v_1, v_1^{q_0{}^2} v_1^{q_2}, v_2{}^y v_1^{q_0 q_2}\rangle$.

We have $C_V(r_1, r_2, r_3, d_1, d_{2+}) = \langle v_1, v_1^{q_0{}^2}\rangle \cup \langle v_1\rangle \{v_1^{k_2}, v_2^{y q_1{}^2 d_2 t_2}\}$ and $v_1^{q_0{}^2} \in C_V(r_0, t_1, d_2, t_2)$; hence, $v_1^{y q_1{}^2} \in v_1^{q_0{}^2} v_1^{q_0 q_2} v_1^{k_2}\langle v_1\rangle$ and $\mathrm{ccl}_T(v_2^{y q_1{}^2}) = v_1{}^i v_1^{q_0{}^2} v_1^{q_0 q_2} \times \{\mathrm{ccl}_{E_2}(v_1^{k_2}) \cup v_1^{q_0{}^5} \cdot \mathrm{ccl}_{E_2}(v_1^{k_2 t_1})\} = v_1{}^i v_1^{q_0{}^2} v_1^{q_0 q_2} v_1^{k_2}\{\langle v_1, v_1^{q_2}, v_1^{q_0 q_2}\rangle \cup v_1^{q_0{}^5} v_1^{q_0 q_1}\langle v_1, v_1^{q_0{}^2} v_1^{q_2}, v_2{}^y v_1^{q_0 q_2}\rangle\} = v_1^{q_0{}^2} v_1^{q_0 q_2} v_1^{k_2}\{\langle v_1, v_1^{q_2}, v_1^{q_0 q_2}\rangle \cup v_2{}^y v_1^{q_0 q_1}\langle v_1, v_1^{q_0{}^2} v_1^{q_2}, v_2{}^y v_1^{q_0 q_2}\rangle\}$, where $i \in \{0, 1\}$.

(e) We have $V_0 = C_{V_0}(r_0)\langle v_1^{k_2}\rangle = C_{V_0}(r_0)\langle v_1^{y q_1{}^3}\rangle = C_{V_0}(r_0) \cup \mathrm{ccl}_T(v_1^{k_2}) \cup \mathrm{ccl}_T(v_2^{y q_1{}^2}) \cup \mathrm{ccl}_T(v_2^{y q_1{}^3})$. Thus, by (1.6) and part (d), $v_1^{q_0} v_1^{k_2} \in v_2^{y q_1{}^3 t_2 d_{2+}}\langle v_1\rangle = v_1^{q_0{}^2} v_1^{q_2} v_1^{q_0 q_2} v_2^{y q_1{}^3}\langle v_1\rangle$ and $v_1^{q_0} v_2^{y q_1{}^2} \in v_2^{y q_1{}^3 d_2}\langle v_1\rangle = v_1^{q_2} v_2^{y q_1{}^3}\langle v_1\rangle$.

Since $\mathrm{ccl}_T(v_1^{y q_1{}^3}) = \mathrm{ccl}_X(v_1^{y q_1{}^3})$ with $X = \langle r_0, d_2, d_{2+}, t_1, t_2\rangle$, we easily verify the statements of (e) by making use of (1.9) and the results of part (d).

(f) To verify the assertions we note that $\mathrm{ccl}_T(v_2) = \mathrm{ccl}_X(v_2)$ with $X = \langle r_1, r_3, d_1, d_{1+}\rangle$, $[V, t_2] = \langle v_1^{q_0{}^3}, v_1^{q_2} v_1^{q_0 q_2}\rangle \cup \mathrm{ccl}_{\langle d_1, d_{1+}\rangle}(v_1 v_2)$, and $[V, d_2] = \langle v_1^{q_0{}^2}, v_1^{q_2}\rangle \cup \mathrm{ccl}_{\langle r_1, r_3\rangle}(v_1 v_2)$ and make use of (1.9) and the results obtained so far in parts (a)–(e). Q.E.D.

(1.11) We already know that $V_1 = \{1, v_1 v_2^{q_1{}^i} \mid 0 \leqslant i \leqslant 6\}$ and $V_1 K_1 / O(K_1) \cong \mathrm{Hol}(E(2^3))$. Hence, $V_1 K_{12}$ is an S_2-subgroup of $V_1 K_1$ and it is isomorphic with S_2-subgroups of $GL_4(2)$.

Furthermore, w.l.o.g. the following coordinations hold:

$$v_1 v_2^{q_1{}^4} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$v_1 v_2^{q_1{}^6} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad d_{2+} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$v_1v_2 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \qquad d_2 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \qquad t_2 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Proof. We may and do assume that $O(K_1) = 1$. Thus, $C_{K_1}(v_1v_2) = C_{K_1}(v_2) = \langle d_2, t_2, a_3, d_{2+}\rangle \cong \Sigma_4$. Note that $d_2^{q_1^j}$ and $d_{2+}^{q_1^j}$ are not contained in $N_L(E_2)$ if $j \in \{2, 3, 4, 6\}$. As $d_2^{q_1} = a_3d_{2+}$, $d_2^{q_1^5} = a_3^2d_{2+}$, $d_{2+}^{q_1} = d_2t_2$, and $d_{2+}^{q_1^5} = t_2d_{2+}a_3$, we get $C_{V_1}(d_2) = C_{V_1}(d_{2+}) = \{1, v_1v_2, v_1v_2^{q_1^2}, v_1v_2^{q_1^6}\}$. Since $t_2^{q_1^2} = d_2d_{2+}$ and $t_2^{q_1^3} = d_2t_2a_3d_{2+}$, we also have $C_{V_1}(t_2) = \{1, v_1v_2, v_1v_2^{q_1^4}, v_1v_2^{q_1^5}\}$. Note that $q_1d_{2+}q_1^2 = d_2d_{2+}t_2$ and thus $[v_1v_2^{q_1}, d_{2+}] = [v_1v_2^{q_1^4}, d_{2+}] = (v_1v_2)^{q_1^5} \times (v_1v_2)^{q_1} = v_1v_2^{q_1^6}$. As t_2 acts on both, $C_{V_1}(d_2)$ and $C_{V_1}(t_2)$, the assertions follow. Q.E.D.

(1.12) Let $N = N_{VL}(E)$.

(a) E is the only subgroup of VT isomorphic to $Ex(2^{13})$.

(b) In E there are exactly $2 \cdot 3^3 \cdot 7 \cdot 11$ noncentral involutions and $2^6 \cdot 3^2 \cdot 7$ elements of order 4.

(c) The following table presents the N-orbits in $E - \langle v_1\rangle$.

Representative of an N-orbit	$o(x)$	Further conjugates	$\lvert \mathrm{ccl}_N(x)\rvert$
$v_1^{q_0}$	2		$2 \cdot 3 \cdot 7$
v_2^{y}	2		$2 \cdot 3 \cdot 7 \cdot 2$
r_0	2	$v_1^{q_0}r_0$	$2 \cdot 3 \cdot 7 \cdot 2^2$
$v_2^{y}r_0$	2	$v_1^{q_2}r_0$, $v_1^{q_0q_2}r_0$	$2 \cdot 3 \cdot 7 \cdot 2^2 \cdot 3$
$v_1^{q_0q_1}r_0$	2		$2 \cdot 3 \cdot 7 \cdot 2^2$
$v_2^{yq_1}r_0$	2	$v_2^{yk_3}r_0$	$2 \cdot 3 \cdot 7 \cdot 2^2 \cdot 3$
r_1r_2	2		$2 \cdot 3 \cdot 7 \cdot 2^4$
$v_2^{y}r_1r_2$	2	$v_1^{q_0}r_1r_2$, $v_2^{yq_1t_1}r_1r_2$	$2 \cdot 3 \cdot 7 \cdot 2^4 \cdot 3$
$v_1^{k_2}r_0$	4	$v_2^{yq_1^2}r_0$	$2 \cdot 3 \cdot 7 \cdot 2^5$
$v_1^{q_0q_1}r_1r_2$	4	$v_2^{yq_1}r_1r_2$, $v_2^{yq_1^2}r_1r_2$, $v_2^{yk_3}r_1r_2$	$2 \cdot 3 \cdot 7 \cdot 2^4 \cdot 3$
$v_2^{yq_1^3}r_1r_2$	4		$2 \cdot 3 \cdot 7 \cdot 2^4$

Proof. (a) Let X be a subgroup of VT with $X \simeq Ex(2^{13})$. As V and E are normal in VT, we have $1 \neq V \cap X \lhd X$ and $1 \neq E \cap X \trianglelefteq X$. So $Z(X) \leqslant V \cap E = V_0$ and $X/(X \cap V)$ is elementary Abelian. Since T has 2-rank six, we get $X_0 = X \cap V \simeq E(2^7)$ and $VX = VR_j$ with a suitable $j \in \{1, 2\}$. Clearly, $Z(X) < X_0$ and so $V\langle r\rangle \cap X$ is a normal subgroup of X of order divisible by 2^8 for any $r \in (R_1 \cap R_2)^\#$; hence, $X' = (X \cap V\langle r\rangle)' < [V, r]$. By (1.9), $X' = Z(X) = \langle v_1\rangle$.

Let $(e_1, ..., e_6) = (r_1r_2, r_0r_1r_2, r_0r_3, r_0r_1r_3, r_0d_{j+}, r_2d_jd_{j+})$; then $R_j = \langle e_k \mid 1 \leqslant k \leqslant 6\rangle$ and $e_k \in \mathrm{ccl}_{N_L(R_j)}(r_1r_2)$ for each $k \in \{1, 2, ..., 6\}$.

Since $X \simeq Ex(2^{13})$, we have $X = X_0R_j^*$, where $R_j^* = \langle w_ke_k \mid 1 \leqslant k \leqslant 6\rangle$ with suitable $w_k \in V$ and where $\langle v_1\rangle R_j^*$ is a maximal Abelian normal subgroup of X. Now put $R^* = \langle w_ke_k \mid 1 \leqslant k \leqslant 4\rangle$; as $\langle v_1\rangle = [X_0, R^*]$ and $|C_{R^*}(v)| \leqslant 2^2$ for any v of $V - V_0$, we conclude that $X_0 = V_0$. Moreover, $\langle v_1\rangle = C_{V_0}(R_j^*) = C_{V_0}(R_j)$; hence $j = 1$. We have proved so far that $X = V_0R_1^*$.

As $(w_ke_k)^2 = [w_k, e_k] \in \langle v_1\rangle = [V_0, e_k]$ and $C_V(e_k) < V_0$, we finally get $w_k \in V_0$ for $k \in \{1, 2, ..., 6\}$. Thus, $X = V_0R_1 = E$.

(b) Considering E as a central product of six quaternion groups of order 8 the verification of these statements will be a trivial counting exercise.

(c) Clearly, $N = N_{VL}(E) = V \cdot N_L(R_1) = O(N_L(R_1))\, EV_1K_1\langle a_1, t_1\rangle$. Since $C_L(v_1) = O(L)\, N_L(R_1)$ and $O(N_L(R_1)) = O(N) \leqslant O(L) \cap C(E)$, we may and do assume that $O(L) = 1$ and thus $N = V \cdot N_L(R_1) = V \cdot C_L(v_1)$ as well as $|N| = 2^{20} \cdot 3^2 \cdot 7$.

As $C_N(x) = C_V(x) \cdot C_L(v_1, x)$ for $x \in (V_0 \cup R_1)$, we obviously have $|\mathrm{ccl}_N(v_1^{q_0})| = 2 \cdot 3 \cdot 7$, $|\mathrm{ccl}_N(v_2{}^y)| = 2^2 \cdot 3 \cdot 7$, $|\mathrm{ccl}_N(r_0)| = 2^3 \cdot 3 \cdot 7$, and $|\mathrm{ccl}_N(r_1r_2)| = 2^5 \cdot 3 \cdot 7$.

Let $\mathscr{M}_0$ denote the set of involutions of E which are not N-conjugates of v_1, $v_1^{q_0}$, $v_2{}^y$, r_0, and r_1r_2 and let $\mathscr{M}_1$ denote the set of elements of order 4 of E. Hence, $|\mathscr{M}_0| = 2 \cdot 3 \cdot 7 \cdot 76$ and $|\mathscr{M}_1| = 2^6 \cdot 3^2 \cdot 7$.

Since $N_L(R_1) = C_L(v_1)$ and $\mathscr{R}_0 = \{v_1^{q_2}r_0, v_1^{q_0q_1}r_0, v_2{}^yr_0, v_2^{yq_1}r_0, v_1^{q_0}r_1r_2, v_2{}^yr_1r_2, v_2^{yq_1t_1}r_1r_2\} \subseteq \mathscr{M}_0$, we calculate the following:

$$C_L(v_1, v_1^{q_2}r_0) = R_2\langle d_{1+}, t_2\rangle,$$

$$C_L(v_1, v_1^{q_0q_1}r_0) = E_1\langle r_3, d_{1+}, a_3, d_{2+}\rangle,$$

$$C_L(v_1, v_2{}^yr_0) = E_2\langle r_1, r_3d_1, d_{1+}, t_1d_{2+}\rangle,$$

$$C_L(v_1, v_2^{yq_1}r_0) = C_T(v_2^{yq_1}) \cdot \langle d_2^{q_1}\rangle,$$

$$C_L(v_1, v_1^{q_0}r_1r_2) = \langle r_0, r_1, r_2, d_1, d_{1+}, d_2, t_1t_2\rangle,$$

$$C_L(v_1, v_2{}^yr_0) = \langle r_0, r_1, r_2, r_3d_1, d_{1+}, d_2, d_{2+}t_1t_2\rangle,$$

$$C_L(v_1, v_2^{yq_1t_1}r_1r_2) = \langle r_0, r_1r_2, r_1r_3, r_1d_1, r_1d_{1+}, d_2t_1t_2\rangle.$$

Obviously, $\mathscr{R}_0$ is a minimal set of representatives of the $N_L(R_1)$-orbits in $\mathscr{M}_0$; thus we have $(N_L(R_1)!\,\mathscr{M}_0) = 2^2 \cdot 3^2 \cdot 7v_1^{q_2}r_0 + 2^3 \cdot 3 \cdot 7v_1^{q_0q_1}r_0 + 2^2 \cdot 3^2 \cdot 7v_2{}^yr_0 + 2^3 \cdot 3^2 \cdot 7v_1^{yq_1}r_0 + 2^3 \cdot 3^2 \cdot 7v_2{}^yr_1r_2 + 2^3 \cdot 3^2 \cdot 7v_1^{q_0}r_1r_2 + 2^4 \cdot 3^2 \cdot 7v_2^{yq_1t_1}r_1r_2$.

As r_0 and r_1r_2 are representatives of the two $N_L(R_1)$-classes in R_1, $vr_0 \notin \mathrm{ccl}_N(wr_1r_2)$ for any elements v and w of V.

Recall $[V, r_0] = C_V(E_1E_2)$, $\mathrm{ccl}_T(v_1^{q_0q_1}) = v_1^{q_0q_1} \cdot C_V(E_1E_2)$, $\mathrm{ccl}_T(v_1^{yq_1}) = v_1^{yq_1} \cdot C_V(r_0, r_1, r_2)$, and $v_2{}^y = v_1^{q_0}v_1^{q_2}$; hence, $\mathrm{ccl}_N(v_1^{q_2}r_0) = \mathrm{ccl}_N(v_2{}^yr_0)$, $\mathrm{ccl}_N(v_1^{q_0q_1}r_0) = \mathrm{ccl}_{N_L(R_1)}(v_1^{q_0q_1}r_0)$, and $\mathrm{ccl}_N(v_2^{yq_1}r_0) = \mathrm{ccl}_{N_L(R_1)}(v_1^{yq_1}r_0)$.

Moreover, we have $[V, r_1r_2] = C_V(R_2)\,\langle v_2^{yq_1}\rangle$ and $v_2^{yq_1t_1} = v_2^{yt_1q_1} = v_1^{q_0{}^2q_1}v_2^{yq_1} = v_1v_1^{q_0{}^2}v_1^{q_0}v_2^{yq_1}$. Thus, there exist elements v and w of V such that $(v_2{}^yr_1r_2)^v = v_1^{q_2}v_2{}^yr_1r_2 = v_1^{q_0}r_1r_2$ and $(v_2^{yq_1t_1}r_1r_2)^w = v_1v_1^{q_0{}^2}v_1^{q_0}r_1r_2 \in \mathrm{ccl}_V(v_1^{q_0}r_1r_2)$.

We have proved that $(N!\,\mathscr{M}_0) = 2^3 \cdot 3^2 \cdot 7v_1^{q_0}r_0 + 2^3 \cdot 3 \cdot 7v_1^{q_0q_1}r_0 + 2^3 \cdot 3^2 \cdot 7v_1^{yq_1}r_0 + 2^5 \cdot 3^2 \cdot 7v_1^{q_0}r_1r_2$.

Clearly, $\mathscr{R}_1 = \{v_1^{k_2}r_0, v_1^{yq_1{}^2}r_0, v_1^{q_0q_1}r_1r_2, v_2^{ya_1}r_1r_2, v_2^{yq_1{}^3}r_1r_2\} \subseteq \mathscr{M}_1$. We have

$$C_L(v_1, v_1^{k_2}r_0) = \langle r_1, r_2, r_3, d_1, d_{1+}, a_3, d_{2+}\rangle,$$

$$C_L(v_1, v_2^{yq_1{}^2}r_0) = \langle r_0r_1, r_2, r_3, r_0d_1, r_0d_{1+}, d_2d_{2+}\rangle,$$

$$C_L(v_1, v_1^{q_0q_1}r_1r_2) = \langle r_0, r_1, r_3, d_1, d_{1+}, d_{2+}\rangle,$$

$$C_L(v_1, v_2^{ya_1}r_1r_2) = \langle r_0, r_2, r_1r_3, r_1d_{1+}, d_1, d_2\rangle,$$

$$C_L(v_1, v_2^{yq_1{}^3}r_1r_2) = \langle r_0r_1, r_0r_3, r_2, d_1, r_0d_{1+}, d_{2+}a_1a_2\rangle.$$

As $\mathscr{R}_1$ is a minimal set of representatives of the $N_L(R_1)$-orbits in $\mathscr{M}_1$, we have $(N_L(R_1)!\,\mathscr{M}_1) = 2^4 \cdot 3 \cdot 7v_1^{k_2}r_0 + 2^4 \cdot 3^2 \cdot 7v_1^{yq_1{}^2}r_0 + 2^4 \cdot 3^2 \cdot 7v_1^{q_0q_1}r_1r_2 + 2^4 \cdot 3^2 \cdot 7v_2^{ya_1}r_1r_2 + 2^5 \cdot 3 \cdot 7v_2^{yq_1{}^3}r_1r_2$.

Since, $v_1^{q_0{}^2}v_1^{k_2} \in \mathrm{ccl}_T(v_2^{yq_1{}^2})$, we get $v_1^{k_2}r_0 \in \mathrm{ccl}_{VT}(v_2^{yq_1{}^2}r_0)$ and $\mathrm{ccl}_N(v_1^{k_2}r_0) = \mathrm{ccl}_{N_L(R_1)}(v_1^{k_2}r_0) \cup \mathrm{ccl}_{N_L(R_1)}(v_2^{yq_1{}^2}r_0)$.

Clearly, $\langle r_0r_1r_2, r_0r_3\rangle \lhd T_1$ and $C_V(r_0r_1r_2, r_0r_3) = [V, r_1r_2]\,\langle v_2^{yq_1{}^3}\rangle$. Thus, by (1.6), $\mathrm{ccl}_{T_1}(v_2^{yq_1{}^3}) = v_2^{yq_1{}^3}[V, r_1r_2]$. Therefore we get $\mathrm{ccl}_N(v_2^{yq_1{}^3}) = \mathrm{ccl}_{N_L(R_1)}(v_2^{yq_1{}^3})$.

We know that there exists an element $v \in V$ such that $[v, r_1r_2] = v_1^{q_2}$; so $(v_1^{q_0q_1}r_1r_2)^{t_1t_2v} = (v_1v_1^{q_0}v_1^{q_0q_1}r_1r_2)^v = v_1v_2{}^yv_1^{q_0q_1}r_1r_2 = v_2^{yk_3}r_1r_2$.

This finally proves $(N!\,\mathscr{M}_1) = 2^6 \cdot 3 \cdot 7v_1^{k_2}r_0 + 2^5 \cdot 3^2 \cdot 7v_1^{q_0q_1}r_1r_2 + 2^5 \cdot 3 \cdot 7v_2^{yq_1{}^3}r_1r_2$.

To prove that $v_1^{q_0q_2}r_0 \in \mathrm{ccl}_{N_L(R_1)}(v_1^{q_2}r_0)$ and $v_2^{yk_3}r_0 \in \mathrm{ccl}_{N_L(R_1)}(v_2^{yq_1}r_0)$ we only note that $C_L(v_1, v_1^{q_0q_2}r_0) = E_2\langle r_3, r_1d_1, d_{1+}, a_3d_{2+}\rangle$ and $k_3t_2q_1 \in N_L(R_1) \cap H_0 = C_L(v_1, r_0)$. As $C_{H_1}(v_2^{yq_1{}^2}) \leqslant R_1R_2$ and $k_3t_1d_{2+}a_1 \in H_1$, the elements $v_2^{yq_1{}^2}r_1r_2$ and $v_2^{yk_3}r_1r_2$ are contained in $\mathrm{ccl}_{N_L(R_1)}(v_2^{ya_1}r_1r_2)$. Q.E.D.

(1.13) Put $W = C_V(E_2) \cdot E_2$ and $V_2 = \langle \mathrm{ccl}_{A_2}(v_1v_1^{q_0q_1})\rangle$; then the following hold:

(a) $E(2^{10}) \cong W = C_{VT}(W) \lhd VT$ and $N_G(V) \cap N_G(W) = V \cdot E_2 \cdot A_2$.

(b) $(VE_2A_2!\,V) = 35v_1 + 28v_2 + 120v_1^{q_0q_1} + 840v_2^{yk_3}$ with $(A_2!\,\mathrm{ccl}_L(v_1)) = 35v_1 + 105v_1^{q_0q_1} + 15v_1v_1^{q_0q_1}$ and $(VE_2A_2!\,W) = 35v_1 + 28v_2 + 120r_0 + 840v_1^{q_2}r_0$ as well as $(A_2!\,W) = 35v_1 + 28v_2 + 105v_1r_0 + 15r_0 + 420v_1^{q_2}r_0 + 420v_2{}^yr_0$.

(c) $V_2 \cong E(2^4)$ and $V = C_V(E_2) \times V_2$; moreover, $C_{V_2}(r_0) = \langle v_1v_1^{q_0q_1}\rangle$, $C_{V_2}(r_2) = \langle v_1v_1^{k_2}\rangle$, $C_{V_2}(d_2) = \langle v_1v_2^{q_1{}^6}\rangle$, $C_{V_2}(t_2) = \langle v_1v_2^{q_1{}^5}\rangle$.

(d) Let $X \in \{V, W\}$ and Y be a nontrivial proper A_2-invariant subgroup of X; then $Y \in \{C_V(E_2), E_2, V_2\}$.

Proof. As $O(A_2) \leqslant O(L) \cap C(VE_2)$, we may assume that $O(L) = 1$. The assertions of (a) are trivial.

(b) We know already that $v_1^{q_0q_1r_2} = v_1v_1^{q_0q_1}$ and $(A_2!\,C_V(E_2)) = 35v_1 + 28v_2$. Since $A_2 \cap C(v_1^{q_0q_1}) = \langle r_1, d_1, t_1\rangle \cdot \langle r_3, d_{1+}, a_3, d_{2+}\rangle$ and $A_2 \cap C(v_1v_1^{q_0q_1}) = \langle r_1, d_1, t_1\rangle \cdot A_0$, we immediately get $(A_2!\,\mathrm{ccl}_L(v_1)) = 35v_1 + 105v_1^{q_0q_1} + 15v_1v_1^{q_0q_1}$. Moreover, as $Z(VT) = \langle v_1\rangle$, $(E_2A_2!\,\mathrm{ccl}_L(v_1)) = 35v_1 + 120v_1^{q_0q_1}$.

As $C_{E_2}(v_2^{yk_3}) = \langle r_0\rangle$, $C_{E_1E_2}(v_2^{yk_3}) = \langle r_0, r_1, r_2d_1, d_2t_1\rangle \cong E(2^4)$, $C_T(v_2^{yk_3}) = C_{E_1E_2}(v_2^{yk_3})\,\langle r_3d_1, d_{1+}, t_1d_{2+}\rangle$, and $|\,\mathrm{ccl}_{E_2A_2}(v_2^{yk_3})| \leqslant 868 - 28 = 840$, we have $|\,C_{E_2A_2}(v_2^{yk_3})| = |\,C_{H_0}(v_2^{yk_3})| = 2^7 \cdot 3$. Thus, $|\,\mathrm{ccl}_{E_2A_2}(v_2^{yk_3})| = 2^3 \cdot 3 \cdot 5 \cdot 7 = 840$.
Clearly, $C_{VE_2A_2}(r_0) = C_V(r_0) \cdot H_0$, $C_{A_2}(r_0) = \langle r_1, d_1, t_1\rangle \cdot A_0$, and $C_{A_2}(v_1r_0) = \langle r_1, d_1, t_1\rangle \cdot \langle r_3, d_{1+}, a_3, d_{2+}\rangle \cong E(2^3) \cdot \Sigma_4$. Furthermore, $C_{A_2}(v_1^{q_2}r_0) = N_L(R_1)^{q_2} \cap H_0 \cap A_2 = \langle r_1\rangle \times \langle r_3, d_{2+}, q_0{}^5a_3{}^2, d_{1+}\rangle \cong Z_2 \times \Sigma_4$, $v_2{}^yr_0 \notin \mathrm{ccl}_{A_2}(v_1^{q_2}r_0)$ and $v_2{}^yr_0 \in \mathrm{ccl}_V(v_1^{q_2}r_0)$, and $|\,C_{A_2}(v_2{}^yr_0)| = |\,B^y \cap H_0\,| \leqslant 2^4 \cdot 3$. From that we easily deduce the required results.

(c) Put $v = v_1v_1^{q_0q_1} = v_1^{q_0q_1r_2}$; then $C_{E_2}(v) = \langle r_0\rangle$. Moreover, we have $C_{A_2}(v) = C_{A_2}(r_0) = \langle r_1, d_1, t_1\rangle \cdot A_0$. Thus, $C_{E_2}(v^a) = \langle r_0{}^a\rangle$ and $C(r_0{}^a) \cap \mathrm{ccl}_{A_2}(v) = \{v^a\}$ for any $a \in A_2$.

As $a_1 \in A_2$ with $a_1\colon r_0 \to r_0r_2 \to r_2$, we calculate $v^{a_1{}^2} = v_1^{k_2r_0} = v_1v_1^{k_2}$ and $v^{a_1} = v_1^{k_2r_0t_1} = v_1v_1^{k_2t_1} = v_1v_1^{q_0q_1}v_1v_1^{k_2} = v \cdot v^{a_1{}^2}$. Since $\langle v\rangle = C_V(q_0)$, we have $\mathrm{ccl}_{A_2}(v) = \{v\} \cup \langle v\rangle \cdot \mathrm{ccl}_{\langle q_0\rangle}(v^{a_1})$ and so $|\,V_2\,| \leqslant 2^8$.
Application of a theorem of Maschke yields $V_2 \cong E(2^4)$. So $V = C_V(E_2) \times V_2$, because A_2 acts irreducibly on both, $C_V(E_2)$ and V_2.
As $q_0\colon r_2 \to r_2t_2 \to d_2 \to t_2$, we calculate $v_1^{k_2r_0q_0{}^2} = v_1^{b_1q_1{}^6} = v_1v_2^{q_1{}^6}$ and $v_1^{k_2r_0q_0{}^3} = v_1^{b_1q_1{}^5} = v_1v_2^{q_1{}^5}$. Now the statements follow by facts proved above.

(d) Let $X \in \{V, W\}$ and Y be a subgroup of X such that $1 \neq Y \neq X$ and $[Y, A_2] \leqslant Y$.

If $Y \cap C_V(E_2) \neq 1$, then $Y = C_V(E_2)$, because A_2 acts irreducibly on $C_V(E_2)$ and $X/C_V(E_2)$. So we assume that $Y \cap C_V(E_2) = 1$. Obviously, $|\,Y\,| = 2^4$ and $Y^{\#} = \mathrm{ccl}_{A_2}(x)$ with a suitable element x of X. Hence, by the results of part (b), $Y \in \{E_2, V_2\}$. Q.E.D.

Notation. Throughout the rest of this paper we use the following notation:

$$V_2 = \langle \mathrm{ccl}_{A_2}(v_1 v_1^{q_0 a_1}) \rangle,$$

$$E_2^* = \langle v_1 v_1^{q_0 a_1} r_0 , v_1 v_1^{k_2} r_2 , v_1 v_2^{q_1^6} d_2 , v_1 v_2^{q_1^5} t_2 \rangle,$$

$$W = C_V(E_2) \cdot E_2 \qquad \text{and} \qquad W^* = C_V(E_2)\, E_2^* .$$

(1.14) The following assertions hold:

(a) $E_2^* \cong E(2^4)$, $E(2^{10}) \cong W^* = C_{VT}(W^*) \lhd VT = W^* \cdot T$, and $V \cdot E_2 = W \cdot V_2 = W^* \cdot E_2 = W^* \cdot V_2 = VW$ as well as $N_G(V) \cap N_G(W^*) = VE_2 \cdot A_2 = W^* \cdot E_2 \cdot A_2$.

(b) $(E_2A_2 ! \; W^*) = 35v_1 + 28v_2 + 120 v_1^{q_0 a_1} r_0 + 840 v_2^{yk_3} r_0$ and $(A_2 ! \; \mathrm{ccl}_{E_2A_2}(v_1^{q_0 a_1} r_0)) = 15 v_1 v_1^{q_0 a_1} r_0 + 105 v_1^{q_0 a_1} r_0$.

(c) $C_V(E_2)$ is the only nontrivial A_2-invariant subgroup of W^*.

(d) If X is an elementary Abelian normal subgroup of VT with $2^{10} \mid |X|$, then $X \in \{V, W, W^*\}$.

Furthermore, V, W, and W^* are the only elementary Abelian subgroups of VW of order divisible by 2^{10}.

Proof. (a) Clearly, E contains $v_1 v_1^{q_0 a_1}$ and $v_1 v_1^{k_2}$. Therefore $[v_1 v_1^{q_0 a_1}, r_2] = v_1 = [v_1 v_1^{k_2}, r_0]$ and so $\langle v_1 v_1^{q_0 a_1} r_0 , v_1 v_1^{k_2} r_2 \rangle \cong E(2^2)$. Moreover, by (1.11), $\langle v_1 v_2^{q_1^6} d_2 , v_1 v_2^{q_1^5} t_2 \rangle \cong E(2^2)$.

We have $q_1 \colon v_1 \to v_1$, $v_1^{q_0} \to v_1^{q_0 a_1}$, $v_1^{q_0^2} \to v_1 v_1^{q_0} v_1^{q_0}$, $v_1^{q_0 a_1} \to v_1 v_1^{q_0^2}$. Making frequent use of results in (1.9), (1.10), (1.11), and (1.13) we get the following:

$$[r_0 , v_1 v_2^{q_1^6}] = [r_1 d_1 , v_2]^{q_1^6} = v_1^{q_0^2} = [d_2 , v_1 v_1^{q_0 a_1}],$$

$$[r_2 , v_1 v_2^{q_1^6}] = [r_0 , v_1 v_2^{q_1^6}]^{a_1^2} = [d_2 , v_1 v_1^{q_0 a_1}]^{a_1^2} = [d_2 , v_1 v_1^{k_2}],$$

$$[r_0 , v_1 v_2^{q_1^5}] = [r_0 r_1 , v_2]^{q_1^5} = v_1^{q_0^2 q_1^5} = v_1 v_1^{q_0} = [t_2 , v_1 v_1^{q_0 a_1}],$$

$$[r_2 , v_1 v_2^{q_1^5}] = [r_0 , v_1 v_2^{q_1^5}]^{a_1^2} = [t_2 , v_1 v_1^{q_0 a_1}]^{a_1^2} = [t_2 , v_1 v_1^{k_2}].$$

Therefore $E_2^* \cong E(2^4)$.

The other assertions of (a) are trivial.

(b) Since $VE_2A_2 = W^* \cdot E_2A_2$, the statements follow by results proved in (1.13) (b).

(c) Suppose X is a nontrivial A_2-invariant subgroup of W^*. If $X \cap C_V(E_2) \neq 1$, then $X = C_V(E_2)$. Henceforth we assume that $X \cap C_V(E_2) = 1$.

Since $A_2/O(A_2) \cong \mathrm{Alt}_8$, we get $|X| = 16$ and $|A_2 \colon C_{A_2}(x)| = 15$ for any $x \in X^\#$. By results of (b), $X^\# = \mathrm{ccl}_{A_2}(v_1 v_1^{q_0 a_1} r_0)$. Put $w = v_1 v_1^{q_0 a_1} r_0$; then $\{w^{a_1^i} \mid 0 \leqslant i \leqslant 2\} = \{w, v_1 v_1^{k_2} r_2 , v_1^{q_0 a_1} v_1^{k_2} r_0 r_2\} \subseteq X$ and so $w^{a_1} \cdot w^{a_1^2} = v_1 \cdot w \in X$ which contradicts the fact that $w \in X$ and $X \cap C_V(E_2) = 1$.

(d) Suppose X is an elementary Abelian normal subgroup of VT with $2^{10} \mid |X|$ and $X \notin \{V, W\}$. Let $X_0 = X \cap V$ and $D = \{d \in T \mid \exists\, w \in V - X_0^{\#}\colon wd \in X\}$; obviously D is an elementary Abelian normal subgroup of T containing $Z(T) = \langle r_0 \rangle$. Moreover, $\langle v_1 \rangle \leqslant X_0 \lhd VT$, $VX = V \cdot D$, and $X_0 \leqslant C_V(D) \leqslant C_V(r_0)$.

As $D \cong X/X_0 \leqslant E(2^6)$, $\langle v_1 \rangle = C_V(R_1) < C_V(R_2) \cong E(2^3)$, $D \leqslant C_T(X_0)$, and $C_T(C_V(r_0)) = \langle r_0 \rangle$, we have $|D| \mid 2^5 \mid |X_0| \mid 2^6$.

Now assume that $|X_0| = 2^5$. Thus, $E(2^5) \cong D < R_i$ for a suitable $i \in \{1, 2\}$ and so $\langle r_0, r_1, r_2 \rangle \leqslant D \cap R_1 \cap R_2 \lhd T$. Now we get $E(2^5) \cong X_0 \leqslant C_V(D) \leqslant C_V(r_0, r_1, r_2) \cong E(2^4)$ which is absurd.

Therefore $|X_0| = 2^6$, $2^4 \mid |D| \mid 2^5$, and $|D \cap R_1 \cap R_2| \leqslant 2^2$.

Suppose that $D < R_i$ for some $i \in \{1, 2\}$. As $\langle R_1 \cap R_2, d_i \rangle d_{i+} = \mathrm{ccl}_T(d_{i+}) \cup \mathrm{ccl}_T(r_0 d_{i+}) \cup \mathrm{ccl}_T(r_i d_{i+})$ and as each of these T-classes generates R_i, we get $D \leqslant (R_1 \cap R_2)\langle d_i \rangle$. So again $\langle r_0, r_1, r_2 \rangle \leqslant D \cap R_1 \cap R_2$ which is impossible.

Note that E_1, E_2, R_1, and R_2 are the only maximal elementary Abelian normal subgroups of T and that $C_V(E_1) \cong E(2^4)$. Thus, we have $D = E_2$, $X_0 = C_V(E_2)$, and $X \cong E(2^{10})$.

Let $(e_1, e_2, e_3, e_4) = (r_0, r_2, d_2, t_2)$ and $\langle w_i \rangle = C_{V_2}(e_i)$ for $i \in \{1, 2, 3, 4\}$; then $C_V(e_i) = C_V(E_2) \cdot \langle w_i \rangle$ and by definition of D there exist $y_i \in C_V(e_i) - C_V(E_2)$ with $y_i e_i \in X$. As $C_V(E_2) \leqslant X$, we may assume that $y_i \in \langle w_i \rangle$. Clearly, $1 = [y_i e_i, y_j e_j] = [y_j, e_i] \cdot [y_i, e_j]$ for any $i, j \in \{1, 2, 3, 4\}$.

Suppose there exist elements $k, l \in \{1, 2, 3, 4\}$ such that $k \neq l$ and $[y_k, e_l] = [y_l, e_k] = 1$; then $y_k, y_l \in C_V(e_k, e_l) = C_V(E_2)$. This implies $y_i \in C_V(E_2) \cap \langle w_i \rangle = 1$ for all $i \in \{1, 2, 3, 4\}$. Thus, $E_2 \leqslant X$ and $X = W$ which contradicts the general assumption $X \notin \{V, W\}$.

Consequently, $1 \neq [y_i, e_j] = [y_j, e_i]$ for all $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$. Therefore we have $y_i = w_i$ and so $E_2^* \leqslant X$. This finally proves $X = W^*$.

Moreover, we have proved that V, W, and W^* are the only elementary Abelian subgroups of VE_2 of order divisible by 2^{10}. Q.E.D.

(1.15) We have $(VT)' = V_0 \cdot \langle v_1 v_2, v_1 v_2^{q_1^2} \rangle \cdot T'$, $(VT)'' = C_V(r_0, r_1) \cdot T'' = C_V(r_0, r_1) \cdot \langle r_0 \rangle$, and $(VT)''' = 1$.

Among others VT contains the following characteristic subgroups: $\langle v_1 \rangle$, $\langle v_1, v_1^{q_0^2} \rangle$, $\langle v_1, v_1^{q_0^2}, v_1^{q_0} \rangle$, $\langle v_1, v_1^{q_0^2}, v_1^{q_2} \rangle$, $\langle v_1, v_1^{q_0^2}, v_2^{y} \rangle$, $\langle v_1, v_1^{q_0^2}, v_1^{q_0}, v_1^{q_2} \rangle$, $C_{V_0}(E_2)$, $C_V(E_2)$, VE_2, $VE_2\langle r_1 \rangle$, $VE_2\langle r_1, r_3, d_1 \rangle$, VE_2R_1, VE_2E_1, VE_2R_2, and E.

Proof. Put $T_2 = K_{12} \times \langle t_1 \rangle$; thus $V = V_0 \times V_1$ and $T = R_1 \cdot T_2$. Now let $v = w_0 w_1 \in V$ and $x = rt \in T$ with $w_0 \in V_0$, $w_1 \in V_1$, $r \in R_1$, and $t \in T_2$; then $[v, x] = [w_1, t] \cdot [w_0, t] \cdot [v, r]^t$. Since $[V, R_1] = V_0 \lhd VT$, we get $[V, T] = V_0 \cdot [V_1, T_2] = V_0 \cdot [V_1, K_{12}] = V_0 \cdot \langle v_1 v_2, v_1 v_2^{q_1^2} \rangle$ and $(VT)' = [V, T] \cdot T' = V_0 \cdot \langle v_1 v_2, v_2 v_2^{q_1^2} \rangle \cdot T'$.

Put $Y = [[V, T], T']$, $V_1^* = \langle v_1 v_2, v_1 v_2^{q_1^2} \rangle$, $R_1^* = R_1 \cap T' = (R_1 \cap R_2)\langle d_1 \rangle$; hence, $(VT)' = V_0 \cdot V_1^* \cdot R_1^* \cdot \langle d_2 \rangle$ and $(VT)'' = Y \cdot T'' = Y \cdot \langle r_0 \rangle$.

Let $v = w_0 w_1 \in [V, T]$ and $rd \in T'$ with $w_0 \in V_0$, $w_1 \in V_1^*$, $r \in R_1^*$, and $d \in \langle d_2 \rangle$. We have $[v, rd] = [w_0, d] \cdot [w_0, r] \cdot [w_1, r]^d$. As $[V_0, d_2] = \langle v_1^{q_0^2} v_1^{q_2} \rangle$, $[V_1^*, d_1] = \langle v_1 v_1^{q_0}, v_1^{q_0 q_1} \rangle$, $[V_0, R_1^*] = \langle v_1 \rangle$, and $\langle v_1, v_1^{q_0^2}, v_1^{q_0}, v_1^{q_2}, v_1^{q_0 q_1} \rangle = C_V(r_0, r_1)$, we easily calculate that $Y = C_V(r_0, r_1) [V_1^*, R_1 \cap R_2]$. Because $[V, E_1] \leqslant C_V(E_1) \leqslant C_V(r_0, r_1)$, we even get $Y = C_V(r_0, r_1) [V_1^*, \langle r_2, r_3 \rangle]$.

So we calculate $[r_2, v_2] = 1 = [r_3, v_2^{q_1^6}]$, $[v_2, r_3] = v_1^{q_2}$, and $[v_2^{q_1^6}, r_2] = v_1^{q_0 q_2 q_1^6} = v_1^{q_2}$. Hence, $Y = C_V(r_0, r_1)$ and $(VT)'' = C_V(r_0, r_1) \langle r_0 \rangle$ as well as $(VT)''' = 1$.

Clearly, VE_2 and E are characteristic subgroups of VT. Thus, $X = (VT)' \cap VE_2$ and X' are characteristic subgroups of VT as well.

Put $Y = [V, T]$; then $[Y, d_2] = [V_0, d_2] = \langle v_1^{q_0^2}, v_1^{q_2} \rangle$, $[Y, r_0] = \langle v_1, [v_2^{q_1^6}, r_0] \rangle = \langle v_1, v_1^{q_0^2} \rangle$, and $[Y, r_2] = \langle v_1, [v_2^{q_1^6}, r_2] \rangle = \langle v_1, v_1^{q_2} \rangle$. Note that $[V, e] \leqslant C_V(E_2)$ for any $e \in X \cap E_2 = \langle r_0, r_2, d_2 \rangle$; hence, $X' = (Y \langle r_0, r_2, d_2 \rangle)' = [Y, \langle r_0, r_2, d_2 \rangle] = \langle v_1, v_1^{q_0^2}, v^{q_2} \rangle = C_V(R_2)$.

Therefore $C_{VT}(C_V(R_2)) = VE_2 R_2$ is a characteristic subgroup of VT. Since VE_2 char VT and VT/VE_2 is a 2-group of type Alt_8, the groups $VE_2\langle r_1 \rangle$, $VE_2 \langle r_1, r_3, d_1 \rangle$, $VE_2 R_1$, and $VE_2 E_1$ are characteristic in VT.

As $C_V(E_2) = Z(VE_2)$, $C_V(E_2) \cap E = C_{V_0}(E_2)$, and $Z(VE_1 E_2) = \langle v_1, v_1^{q_0^2}, v_1^{q_0} \rangle$, the required results hold.

(1.16) Lemma. *Let H be a finite group with $V < H$, $N_H(V) = N_G(V)$, and $V \cdot T \in \mathrm{Syl}_2(H)$.*

Then $N_H(VT) = (O(L) \cap C(T)) \times VT$; moreover, one of the following two statements holds:

(a) $H = O(H) N_G(V)$.

(b) *H contains a normal subgroup N such that $O(H) \lhd N$, $VT \in \mathrm{Syl}_2(N)$, and $N/O(H)$ is a non-Abelian simple group; moreover, $H/N \cong N_H(VT)/N_N(VT) \cong (O(L) \cap C(T))/(O(L) \cap C(T) \cap N)$.*

Proof. Let H be a finite group satisfying the assumptions of the lemma. Then $C_H(VT) = (O(L) \cap C(T)) \times \langle v_1 \rangle$ and $Q = O(L) \cap C(T) = O(N_H(VT)) \leqslant O(A_2)$ as well as $N_H(VT) \cap N_G(V) = Q \times VT$.

Recall that VE_2 is a characteristic subgroup of VT containing exactly three elementary Abelian subgroups of order 2^{10}, namely V, W, and W^*. Hence, $N_H(VT)/Q$ is a $\{2, 3\}$-group.

Now suppose that there exists an element x in $N_H(VT) - Q$ with $x^3 \in Q$. Without loss of generality we may assume that x: $V \to W^* \to W \to V$. Furthermore, x acts on VE_2 and $K = VE_2 A_2$. Clearly, $x^3 \in Q \leqslant O(A_2) = O(K) \leqslant C_H(VE_2)$ and $[A_2, x] \leqslant O(K) \cdot VE_2$ as well as $\bar{K} = K/O(K) \cong VE_2 \cdot \mathrm{Alt}_8$.

As V is a completely reducible A_2-module, W^* is the direct product of two nontrivial A_2^*-invariant subgroups. Let $A_x = \{a \in E_2 A_2 \mid \exists\, w \in W^*: wa \in A_2{}^x\}$; then $\mathrm{Alt}_8 \cong \bar{A}_x < \bar{K} = \bar{W}^* \cdot \bar{E}_2 \cdot \bar{A}_x$. Obviously, W^* is the direct product of two nontrivial $\bar{A}_x$-invariant subgroups. As $\bar{E}_2 \bar{A}_x = \bar{E}_2 \bar{A}_2$ and $\bar{A}_x \in \mathrm{ccl}_{\bar{K}}(\bar{A}_2)$,

W^* is the direct product of two nontrivial $\bar{A}_2$-invariant subgroups. But this contradicts (1.14) (c).

We have proved that $C_H(VT) = Q \times \langle v_1 \rangle$ and $N_H(VT) = Q \times VT$.

Obviously, $O_{2'2}(H) \in \{O(H), O(H) \cdot V\}$. If $O_{2'2}(H) = O(H) \cdot V$, then $H = O(H) N_H(V) = O(H) N_G(V)$ by the Frattini argument. So we suppose that $O_{2'2}(H) = O(H)$. Let $\bar{H} = H/O(H)$ and let N be subgroup of H containing $O(H)$ such that $\bar{N}$ is a minimal normal subgroup of $\bar{H}$. Clearly, $O(N) = O(H)$, $VT \in \mathrm{Syl}_2(N)$, and $\bar{N}$ is non-Abelian simple. Making use of the Frattini argument we get $H = NN_H(VT) = NQ$. Hence $H/N \cong Q/(Q \cap N) \cong C_H(VT)/C_N(VT) \cong N_H(VT)/N_H(VT)$. Q.E.D.

2. S_2-Subgroups and Other 2-Locals of G

By our assumption (A1) G contains a subgroup S and an element s such that $|S:VT| = 2$ and $S = VT\langle s \rangle = W^*T\langle s \rangle$.

In what follows let $A_c = \langle O(A_2) \cap C(S), A, T \cap A_2 \rangle$ and let U denote the subgroup of S generated by W^* and the element s; hence, $A_2 = O(A_2)\, A_c$ and $A_c/O(A_c) \cong \mathrm{Alt}_8$.

(2.1) Let $X = \langle VE_2A_2, s \rangle$.

(a) Without loss of generality we may assume that $s^2 = 1$ and thus $X = VE_2A_2 \cdot \langle s \rangle$; moreover, $[A_2, s] \leqslant O(A_2) \cdot C_V(E_2)$ and $C_V(E_2) \cdot \langle s \rangle \cong E(2^7)$.

(b) $O(A_2)\, C_V(E_2)\, \langle s \rangle$ is a normal subgroup of X with S_2-subgroup $F = C_V(E_2)\, \langle s \rangle$; so $X = O(A_2)\, N_X(F)$ and w.l.o.g. we have $N_X(F) = VE_2\langle s \rangle \cdot A_c$ with $O(A_c) = O(A_2) \cap C(s)$.

Furthermore, one of the following two cases holds.

(i) $[A_c, s] = 1$ and
$$(A_c!\,F) = 35v_1 + 28v_2 + 1s + 35v_1s + 28v_2s.$$

(ii) $A_c \cap C(s) = O(A_c)\, A$ and
$$(A_c!\,F) = 35v_1 + 28v_2 + 8s + 56v_1s.$$

(c) We have $V^s = W$, $V_2^s = E_2$, and $(W^*)^s = W^*$; moreover, $V_2 \cap C(e) = \langle e^s \rangle$ for each e of $E_2^\#$.

(d) Clearly, $r_1^s = r_1$ in case (i); in case (ii) we have $r_1^s = v_2^y r_1$.

Proof. (a) Clearly, $N_G(V) \cap N_G(W) = N_G(V) \cap N_G(W^*) = VE_2 \cdot A_2$ and $V^s \in \{W, W^*\}$; hence, s acts on VE_2 and VE_2A_2. As $s^2 \in VT \leqslant VE_2A_2$, we have $X = VE_2A_2\langle s \rangle$.

By (1.15), VE_2E_1 and VE_2R_2 are characteristic subgroups of VT. Thus, $X/O(A_2)\, VE_2 \cong \mathrm{Alt}_8 \times Z_2$ and w.l.o.g. we may assume that $[A_2, s] \leqslant O(A_2) \cdot VE_2$; in particular $s^2 \in VE_2$.

Since $\langle v_1\rangle \leqslant C_V(E_2, s)$ and A_2 acts on both, $C_V(E_2, s)$ and $C_V(E_2)$, we get $C_V(E_2) \leqslant C_{VT}(s)$.

Let $\{Y\} = \{W, W^*\} - \{V^s\}$; hence $Y^s = Y$. Moreover, put $\bar{X} = X/(O(A_2) \cdot Y)$ and $\tilde{X} = X/(O(A_2) \cdot C_V(E_2))$. Note that $\bar{A}_2$ acts on $\overline{VE}_2 = \bar{V}_2$ and on $\bar{V}_2\langle\bar{s}\rangle$; as $\bar{V}_2 \simeq E(2_4)$ and $|\bar{V}_2\langle\bar{s}\rangle| = 2^5$, we have $\bar{V}_2\langle\bar{s}\rangle \simeq E(2^5)$ and w.l.o.g. we may assume that $[\bar{A}_2, \bar{s}] = 1$. The same arguments yield $\tilde{Y}\langle\tilde{s}\rangle \simeq E(2^5)$ and $[\tilde{A}_2, \tilde{s}] = 1$. Hence, $s^2 \in C_V(E_2)$ and $[A_2, s] \leqslant O(A_2) \cdot C_V(E_2)$.

Clearly, $A_2/O(A_2)$ acts on both, $C_V(E_2)$ and $O(A_2)\, C_V(E_2)\,\langle s\rangle/O(A_2)$. This finally proves $s^2 = 1$ and $C_V(E_2)\,\langle s\rangle \simeq E(2^7)$.

(b) Let $F = C_V(E_2)\,\langle s\rangle$. By (a), $O(A_2) \cdot F$ is a normal subgroup of X; the Frattini argument yields $X = O(A_2)\, N_X(F)$, where $O(N_X(F)) = O(A_2) \cap C(s)$ and $O_{2'2}(N_X(F)) = (O(A_2) \cap C(s)) \cdot VE_2\langle s\rangle$. Thus, we may and do assume that $N_X(F) = VE_2\langle s\rangle \cdot A_c$ with $A_c \cap VE_2\langle s\rangle = 1$ and $O(A_c) = O(N_X(F))$.

If $[A_c, s] = 1$, then obviously $(A_c!F) = 35v_1 + 28v_2 + 1s + 35v_1s + 28v_2s$.

Henceforth we suppose that $[A_c, x] \neq 1$ for each element x of $F - C_V(E_2)$. In particular we have $1 \neq |A_c\colon C_{A_c}(s)| \leqslant 64$. Since $O(A_c) \leqslant C_X(s)$, we may assume that $O(A_c) = 1$ and hence, $F_{21} \lesssim A_c \cap C(s) \lesssim \mathrm{Alt}_8$. Considering the structure of Alt_8 we see that $A_c \cap C(s)$ is isomorphic either to $E(2^3) \cdot GL_3(2)$ or to Alt_7. Thus $|\operatorname{ccl}_{A_c}(s)| \in \{15, 8\}$.

Now suppose that $|\operatorname{ccl}_{A_c}(s)| = 15$ and let x be an element of $\mathscr{M} = F - (C_V(E_2) \cup \operatorname{ccl}_{A_c}(s))$. Then $7 \mid |\operatorname{ccl}_{A_c}(x)| \leqslant |\mathscr{M}| = 64 - 15 = 49$. Since Alt_8 contains no subgroups the index of which is an element of $\{7, 14, 21, 42, 49\}$, we get $|\operatorname{ccl}_{A_c}(x)| \in \{28, 35\}$. But that contradicts $|\mathscr{M}| = 49$.

We have proved that $|\operatorname{ccl}_{A_c}(s)| = 8$. Hence, we may assume that $C(s) \cap A_c = A \simeq \mathrm{Alt}_7$.

Note that $C_A(v_2) = B' \simeq \mathrm{Alt}_6$ and that $N_A(\langle x\rangle)$ controls A-fusion of elements of $C_V(E_2, x)$ for any $x \in A$ with $o(x) \in \{5, 7\}$. Thus, $(A!F) = 35v_1 + 7v_2 + 35v_1s + 7v_2s + 21vs$, where $v \in \operatorname{ccl}_{A_2}(v_2) = \operatorname{ccl}_{A_c}(v_2)$. Since $|\operatorname{ccl}_{A_c}(s)| = 8$ and since A_c contains no subgroups with index 21, we get $\operatorname{ccl}_{A_c}(s) = \{s\} \cup \operatorname{ccl}_A(v_2s)$ and $\operatorname{ccl}_{A_c}(v_1s) = \operatorname{ccl}_A(v_1s) \cup \operatorname{ccl}_A(vs)$. So $(A_c!F) = 35v_1 + 28v_2 + 8s + 56v_1s$.

(c) Without loss of generality $O(A_2) = 1$; thus $A_c = A_2$. Note that both, V and V^s contain exactly one complement of $C_V(E_2)$ which is $(C(s) \cap A_c)$-invariant. Since $\operatorname{ccl}_{A_c}(x) = \operatorname{ccl}_A(x)$ for $x \in \langle v_1v_1^{q_0q_1}, r_0\rangle$, we easily see that the claimed results hold.

(d) Suppose $C_{A_c}(s) = A$. Note that $(b_1b_2)^y = t_1d_{2+} \cdot r_1r_3d_1$, $(t_1d_{2+})^y = t_1d_{2+}$, $(a_1a_3)^y = a_1a_3$ and $a_1a_3\colon r_1r_3d_1 \to r_1d_{1+} \to r_1d_{1+} \cdot r_1r_3d_1$. Hence, $\langle r_1r_3d_1, r_1d_{1+}, t_1d_{2+}\rangle \in \mathrm{Syl}_2(A)$ and $1 \neq [s, r_1] \in C_V(E_2, r_1, d_{1+}, r_3d_1, t_1d_{2+}) = \langle v_1, v_2{}^y\rangle$. Moreover, $[s, r_1] = s \cdot s^{r_1} \in \operatorname{ccl}_A(v_2)$. So $[s, r_1] \in v_2{}^y\langle v_1\rangle$. As $b_2t_1 \in C_A(s, r_1, v_2{}^y) - C(v_1)$, we get $[s, r_1] = v_2{}^y$. Q.E.D.

(2.2) The following statements hold.

(a) $E(2^{11}) \simeq U = C_S(U) \lhd S = U \cdot T$ and $VE_2A_2\langle s\rangle = O(A_2)\,(U \cdot E_2 \cdot A_c)$.

(b) U is the only elementary Abelian normal subgroup of S of order divisible by 2^{11}.

(c) We have $U \cdot E_2 \cdot A_c = VE_2A_2\langle s\rangle \cap N_G(U)$ and $(E_2A_c!\, W^*) = 35v_1 + 28v_2 + 120v_1^{q_0a_1}r_0 + 840v_2^{yk_3}r_0$. Moreover, $(E_2A_c!\, U - W^*) = 16s + 560v_1s + 448v_2s$ in case (i) and $(E_2A_c!\, U - W^*) = 128s + 896v_1s$ in case (ii).

Proof. (a) By the results of (2.1) we get $U \cong E(2^{11})$ and $VE_2A_2\langle s\rangle = O(A_2)\,(U \cdot E_2 \cdot A_c)$. Since $T \in \mathrm{Syl}_2(E_2A_c)$, we have $S = U \cdot T$ and $C_S(U) = U \cdot C_T(U) = U \cdot C_{E_2}(E_2^*, s) = U$.

(b) Suppose X is an elementary Abelian normal subgroup of S such that $2^{11} \mid |X|$. Then $2^{10} \mid |X \cap VT|$ and so, by (1.14), $X \cap VT \in \{V, W, W^*\}$. Since $X \cap VT \lhd S$, we get $VT \cap X = W^*$ and $X \leqslant C_S(W^*) = U \cdot C_T(W^*) = U$. Hence, $X = U$.

(c) As $O(A_c) = O(A_2) \cap C(s) \leqslant C_L(UE_2)$ and $A_2 = O(A_2)\,A_c$, we have $VE_2A_2\langle s\rangle \cap N_G(U) = U \cdot E_2 \cdot A_c$.

Furthermore we may and do assume that $O(A_2) = 1$; thus $A_2 = A_c \cong \mathrm{Alt}_8$. To determine the A_2-orbits on W^* we calculate $|C_{A_2}(v_2^{yk_3t_2^i}r_0)| = 2^4 \cdot 3$ for $i \in \{0, 1\}$. Now suppose that there exists an element x of A_2 with $(v_2^{yk_3t_2}r_0)^x = v_2^{yk_3}r_0$; then $x \in H_0 \cap A_2$ and $yk_3t_2xk_3y \in C_L(v_2) = O(L)\,E_2B$. Since $k_3 = t_2a_2$ and $a_2^{t_2} = a_2{}^2$, we get $a_2{}^2xa_2{}^2 \in O(L)\,E_2B^y \leqslant O(L)\,E_2A_2$. But this is impossible for any $x \in H_0 \cap A_2$. So, by (1.14), $(A_2!\, W^*) = 35v_1 + 28v_2 + 15v_1v_1^{q_0a_1}r_0 + 105v_1^{q_0a_1}r_0 + 420v_2^{yk_3}r_0 + 420v_2^{yk_3t_2}r_0$. In particular $(E_2A_2!\, W^*) = 35v_1 + 28v_2 + 120v_1^{q_0a_1}r_0 + 840v_2^{yk_3}r_0$.

Now suppose that case (i) holds. Then $C(s) \cap E_2A_2 = A_c = A_2$. Furthermore, $(v_1s)^{r_0} = v_1^{q_0a_1}r_0s$, $(v_2^{yk_3t_2}r_0s)^{r_0} = v_2^{yk_3t_2}v_1v_1^{q_0a_1}s = v_1v_1^{q_2}s \in \mathrm{ccl}_{A_2}(v_1s)$, and $(v_2^{yk_3}r_0s)^{r_0} = v_2^{yk_3}v_1v_1^{q_0a_1}s = v_2{}^ys \in \mathrm{ccl}_{A_2}(v_2s)$. As $v_1s \notin \mathrm{ccl}_{E_2A_2}(v_2s)$, we finally get $(E_2A_2!\, U - W^*) = 16s + 560v_1s + 448v_2s$.

Henceforth we suppose that case (ii) holds. Let $x \in W^*$; then $xs \in \mathrm{ccl}_{E_2A}(v's)$ with a suitable $v' \in C_V(E_2)$, because A acts transitively on $(W^*/C_V(E_2)^{\#}$ and $(v_1v_1^{q_0a_1}r_0s)^{r_0} = s$. If $v \in C_V(E_2)$, $e \in E_2$, and $a \in A_2$ with $(vs)^{ea} = vs$, then $vs = e^a \cdot v^a \cdot e^{sa} \cdot [a, s] \cdot s$ and thus $e = 1$; hence, $C(vs) \cap E_2A_2 = C(vs) \cap A_2$ and $|\mathrm{ccl}_{E_2A_2}(vs)| = 16 \cdot |\mathrm{ccl}_{A_2}(vs)|$.

Making use of (2.1) we get $|\mathrm{ccl}_{E_2A_2}(s)| = 16 \cdot 8 = 128$ and $|\mathrm{ccl}_{E_2A_2}(v_1s)| = 16 \cdot 56 = 896$. Since $|U - W^*| = 1024$, we finally conclude that $(E_2A_2!\, U - W^*) = 128s + 896v_1s$. Q.E.D.

(2.3) The elements r_0, r_1, r_1r_2, and $r_1r_3d_1$ are representatives of the four conjugacy classes of involutions of both, E_2A_2 and E_2A_c. Moreover we have

$$\begin{aligned} C_U(r_0) &= C_V(E_2)\,\langle v_1^{q_0a_1}r_0\rangle \cong E(2^7), && \\ C_U(r_1) &= C_{W^*}(r_1)\,\langle s\rangle \cong E(2^8) && \text{in case (i)}, \\ &= C_{W^*}(r_1) && \\ &= C_V(E_2, r_1)\,\langle v_1^{q_0a_1}r_0,\, v_1^{k_2}r_2,\, v_2^{a_1^6}d_2\rangle && \text{in case (ii)}, \end{aligned}$$

$$C_U(r_1r_2) = C_V(E_2, r_1)\langle v_2^{yq_1}r_0, v_1^{k_2}r_2\rangle \cong E(2^6),$$

$$C_U(r_1r_3d_1) = \langle v_1, v_1^{q_0^2}, v_2^{y}, v_1^{q_0q_2}, v_1^{q_0q_1}r_0, v_1^{k_2}r_2, s\rangle \cong E(2^7),$$

$$\{x \in U \mid [x, r_1r_2] \in \langle v_1\rangle\} = C_U(r_1r_2)\langle v_1^{q_0q_2}\rangle \cong E(2^7).$$

Proof. As $T \in \mathrm{Syl}_2(E_2A_2) \cap \mathrm{Syl}_2(E_2A_c)$ and $E_2A_2/O(A_2) \cong E_2A_c/O(A_c) \cong \mathrm{Hol}(E(2^4))$, the elements r_0, r_1, r_1r_2, and $r_1r_3d_1$ are representatives of the four conjugacy classes of involutions in both, E_2A_2 and E_2A_c.

Since $[W^*, r_0] = [V, r_0]$ and $[s, r_0] = v_1v_1^{q_0q_1}r_0$, we get $C_U(r_0) = C_{W^*}(r_0) = C_V(E_2)\langle v_1^{q_0q_1}r_0\rangle \cong E(2^7)$.

Clearly, $C_{W^*}(r_1) = C_V(E_2, r_1)\langle v_1^{q_0q_1}r_0, v_1^{k_2}r_2, v_2^{q_1^6}d_2\rangle \cong E(2^7)$ and $W^* = C_{W^*}(r_1) \times \langle v_1^{q_0q_2}, v_2, v_2^{q_1^5t_2}\rangle$ as well as $[W^*, r_1] = \langle v_1, v_1^{q_0^2}, v_1^{q_0}v_1^{q_0q_1}r_0\rangle$. Obviously, $C_U(r_1) = C_{W^*}(r_1)\langle s\rangle \cong E(2^8)$ in case (i), whereas $[r_1, s] = v_2^{y}$ and thus $C_U(r_1) = C_{W^*}(r_1) \cong E(2^7)$ in case (ii).

As $[W^*, r_1r_2] \leqslant V\langle r_0\rangle$ and $[s, r_1r_2] \in Vr_2$, we get $C_U(r_1r_2) = C_{W^*}(r_1r_2) = C_V(E_2, r_1)\langle v_1^{q_0q_1}v_1^{q_0q_2}r_0, v_1^{k_2}r_2\rangle \cong E(2^6)$. Since $U/C_U(r_1r_2) \cong [U, r_1r_2]$ and $\langle v_1\rangle = Z(S)$, the set $D = \{x \in U \mid [x, r_1r_2] \in \langle v_1\rangle\}$ is a subgroup of U which contains $C_U(r_1r_2)$ with index 2. Hence, $D = C_U(r_1r_2)\langle v_1^{q_0q_2}\rangle$.

Clearly, $[r_1r_3d_1, s] = 1$; so $C_U(r_1r_3d_1) = C_{W^*}(r_1r_3d_1)\langle s\rangle$. Obviously, $C_{W^*}(r_1r_3d_1) \leqslant C_V(E_2)\langle v_1^{q_0q_1}r_0, v_1^{k_2}r_2\rangle$. Since $r_1r_3d_1$ centralizes $\langle v_1^{q_0q_1}, v_1^{k_2}\rangle$, we have $C_U(r_1r_3d_1) = C_V(E_2, r_1r_3d_1) \cdot \langle v_1^{q_0q_1}r_0, v_1^{k_2}r_2, s\rangle = \langle v_1, v_1^{q_0^2}, v_2^{y}, v_1^{q_0q_2}, v_1^{q_0q_1}r_0, v_1^{k_2}r_2, s\rangle \cong E(2^7)$. Q.E.D.

(2.4) The 2-group S has the following properties:

(a) U is the only elementary Abelian subgroup of order divisible by 2^{11} of S.

(b) E is the only extra-special subgroup of order 2^{13} in S.

(c) Among others UE_1, UE_2, UR_1, and UR_2 are characteristic subgroups of S.

(d) If X is an elementary Abelian subgroup of UE_2 of order 2^{10} such that X is not contained in U, then X is an element of $\{V, W\}$.

(e) S is an S_2-subgroup of G.

Proof. (a) Let X be an elementary Abelian subgroup of S with $2^{11} \leqslant |X|$ and $X \neq U$; moreover, let $X_0 = X \cap U$. As S/U has 2-rank 6, we get $2^5 \mid |X_0| \mid 2^8$.

If $|X_0| = 2^5$, then $X_0 \leqslant C_U(R_i) = C_V(R_i)$ for some $i \in \{1, 2\}$; but this is absurd, because $C_V(R_1) < C_V(R_2) \cong E(2^3)$.

If $|X_0| = 2^8$, then w.l.o.g. $X_0 = C_U(r_1)$ because of the results in (2.3). Hence, $X \leqslant C_S(X_0) = U \cdot C_T(X_0) = U \cdot \langle r_1\rangle$. So $2^{10} \leqslant |X_0|$ which is impossible.

Now suppose that $X_0 = X \cap UE_2$. Then $X \leqslant UR$, where R is a complement of $\langle r_0, r_2\rangle$ in R_1; furthermore, $|X_0| = 2^7$ and thus $UX = UR$. Since $\mathrm{ccl}_{\langle d_2, t_2\rangle}(r_1r_3d_1) = r_1r_3d_1 \cdot \langle r_0, r_2\rangle$, we may assume that $r_1r_3d_1 \in R$. So $X_0 =$

$C_U(r_1r_3d_1)$ and $X \leqslant C_S(X_0) = U \cdot C_T(X_0) = U \cdot \langle r_1r_3d_1 \rangle$, which contradicts $|X_0| = 2^7$.

Thus, there exist $e \in E_2^{\#}$ and $u \in U$ such that $ue \in X$; moreover, $E(2^6) \leqslant X_0 \leqslant C_U(e) = E(2^7)$.

Now suppose that $C_V(E_2) \leqslant X_0$. Hence $X \leqslant C_S(C_V(E_2)) = UE_2$. So $UX = UE_2$ and $X_0 = C_U(e) \cong E(2^7)$. But on the other hand we get $X_0 \leqslant C_U(E_2) = C_V(E_2) \cong E(2^6)$. We have derived a contradiction. If $|X_0| = 2^7$ or $X \leqslant UE_2$. we get $C_V(E_2) \leqslant X_0$ in either case; but this is impossible.

We have proved so far that $E(2^6) \cong X_0 < X \cap UE_2 < X$ and $X_0 \cap C_V(E_2) \cong E(2^5)$. Moreover, there exist $\tilde{u} \in U$ and an involution $x \in E_2A_c - E_2$ such that $\tilde{u}x \in X$. As $C_U(E_2, r_1) \cong C_U(E_2, r_1r_3d_1) \cong E(2^4)$, we get $X_0 \cap C_V(E_2) \leqslant C_U(E_2, x) \cong E(2^4)$ which is obviously a contradiction. Therefore the required result holds.

(b) Let $X < S$ with $X \cong Ex(2^{13})$. Since S/E has 2-rank 5, we get $Z(X) = X' = (X \cap E)' = E' = \langle v_1 \rangle$. As $X_0 = X \cap U \geqslant Z(X)$ and $S = U \cdot T$ with $m_2(T) = 6$, we have $E(2^7) \cong X_0 \lhd X$ and $D = \{d \in T \mid \exists\, u \in U - X_0^{\#}: ud \in X\} \cong X/X_0 \cong E(2^6)$. Hence, $D = R_i$ for some $i \in \{1, 2\}$.

Now let $R_i = \langle e_j \mid 1 \leqslant j \leqslant 6 \rangle$ with $(r_1r_2, r_0r_1r_2, r_0r_3, r_0r_1r_3) = (e_1, e_2, e_3, e_4)$. Thus $e_k \in \mathrm{ccl}_{\langle a_1, t_1 \rangle}(e_1)$ for each $k \in \{1, 2, 3, 4\}$. Clearly, $X = X_0R_i^*$, where $R_i^* = \langle w_je_j \mid 1 \leqslant j \leqslant 6 \rangle$ with suitable elements w_j of U and where $\langle v_1 \rangle R_i^*$ is a maximal Abelian normal subgroup of X. Since $E(2^6) \cong C_{X_0}(w_je_j) = C_{X_0}(e_j)$ for each $j \in \{1, 2, ..., 6\}$, we get $X_0 = C_U(e_1)\, C_U(e_2) = U \cap E = C_{V_0}(E_2) \cdot \langle v_1^{q_0q_1}r_0, v_1^{k_2}r_2 \rangle$ by the results of (2.3). Note that $X_0 > C_V(R_2) \cong E(2^3)$ and $C_{X_0}(D) = \langle v_1 \rangle$; hence, $D = R_1$ and we may assume that $(e_5, e_6) = (r_2d_1, r_0d_{1+})$. We have $a_3^2: e_1 \to e_5$, $e_3 \to e_6$ and $(w_je_j)^2 = [w_j, e_j] \in \langle v_1 \rangle$; making use of (2.3) we see that $w_j \in X_0$ for each $j \in \{1, 2, ..., 6\}$. Thus $X = X_0R_1^* = X_0R_1 = E$.

(c) Note that E_1, E_2, R_1, and R_2 are the only maximal elementary Abelian normal subgroups of T. Furthermore, U and E are characteristic subgroups of $S = U \cdot T$. As $|UE_1 \cap E| \neq |UE_2 \cap E|$ and $|UR_1 \cap E| \neq |UR_2 \cap E|$, the claimed results hold.

(d) Let $X < UE_2$ with $E(2^{10}) \cong X \not< U$. Suppose there exists an element $x = we \in W^*E_2$ with $w \in W^*$, $e \in E_2$, and $xs \in X$; then $1 = (xs)^2 = [w, e] \cdot [e, s]$ and so $[w, e] = [e, s] = e^se \in [W^*, E_2] \cap V_2 \cdot e = C_V(E_2) \cap V_2 \cdot e$. This implies $e = 1$ and $xs \in U$. Thus, $X \leqslant C_{UE_2}(xs) = U \cdot C_{E_2}(xs) = U$ which is a contradiction.

We have proved that $X \leqslant W^*E_2 = VE_2$. Hence, by (1.14) (d), $X \in \{V, W\}$.

(e) Suppose there exists an element $x \in G$ with $|S\langle x \rangle : S| = 2$; as U and UE_2 are characteristic subgroups of S, we get $V^x = W$ and so $xs \in S\langle x \rangle \cap N_G(V) = V \cdot T$. Thus $x \in VT\langle s \rangle = S$ which is absurd. This proves $S \in \mathrm{Syl}_2(G)$.

Q.E.D.

(2.5) We have $C_G(U) = U \times O(C_G(U))$; moreover, $N_G(U)$ controls the G-fusion in U.

If $N_G(U) > C_G(U)\, E_2 A_c$, then $N_G(U)/C_G(U)$ is isomorphic to the Mathieu group M_{24}.

Proof. As $C_S(U) = U \lhd S$, $U \in \mathrm{Syl}_2(C_G(U))$ and so $C_G(U) = U \times O(C_G(U))$. Furthermore, $N_G(U)$ controls the G-fusion in U, because U is weakly closed in S with respect to G.

Now let $M = N_G(U)$ and $\overline{M} = M/C_G(U)$ as well as $E(2^4) \cdot \mathrm{Alt}_8 \cong \bar{E}_2 \bar{A}_c < \overline{M}$. Clearly, $2^{10} \cdot 3^2 \cdot 5 \cdot 7 \mid \mid \overline{M} \mid \mid 2^{10} \cdot \mid GL_{11}(2)|_{2'} = 2^{10} \cdot 3^6 \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 17 \cdot 23 \cdot 31^2 \cdot 73 \cdot 89 \cdot 127$.

Obviously $C_{\overline{M}}(\bar{E}_2) = \bar{E}_2 \times O(C_{\overline{M}}(\bar{E}_2))$ and $N_{\overline{M}}(\bar{E}_2) = C_{\overline{M}}(\bar{E}_2)\, \bar{A}_c$. Let Q denote the complete preimage of $O(C_{\overline{M}}(\bar{E}_2))$ in M. Since $U/C_U(r_0) \cong [U, r_0] \cong E(2^4)$ and $U/C_U(r_1 r_2) \cong [U, r_1 r_2] \cong E(2^5)$, we get $\mid C_{\overline{M}}(\bar{r}_0)| \mid 2^{49} \cdot 3^3 \cdot 5 \cdot 7^2$ and $\mid C_{\overline{M}}(\bar{r}_1 \bar{r}_2)| \mid 2^{45} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$ by results of [1]. Thus, $N_{\overline{M}}(\bar{E}_2) = \bar{Q} \times \bar{E}_2 \bar{A}_c$ and $\mid \bar{Q} \mid \mid 3^2 \cdot 5 \cdot 7$.

Suppose that $\bar{Q} \neq 1$. If $q \in Q$ with $o(\bar{q}) \in \{3, 5, 7\}$, then A_c acts on $C_U(q)$ and on $[U, q]$. Since $\mid X \mid \in \{2, 2^6, 2^7, 2^{10}\}$ for any A_c-invariant nontrivial subgroup X of U, application of a theorem of Maschke yields $C_U(q) = \langle s \rangle$, $[U, q] = W^*$, $o(\bar{q}) = 3$, and $\mid \bar{Q} \mid \mid 3^2$ as well as $A_c \leqslant C_M(s)$.

Put $X = \langle \bar{q} \rangle \times \bar{E}_2 \bar{A}_c$; then X acts on W^* and $U - W^*$. Note that $(\bar{E}_2 \bar{A}_c!\ U - W^*) = 16s + 560 v_1 s + 448 v_2 s$. As $C_X(s) = \langle \bar{q} \rangle \times \bar{A}_c$, we have $\mid \mathrm{ccl}_X(s)| = 16$. If $3^3 \mid \mid C_X(v_i s)|$ for any $i \in \{1, 2\}$, then $\langle s, v_i s \rangle \leqslant C_U(q)$ which is impossible; thus, $\mid \mathrm{ccl}_X(v_i s)|_3 = 3$ for each $i \in \{1, 2\}$. But on the other side, $560 = 2^4 \cdot 5 \cdot 7$, $448 = 2^6 \cdot 7$, and $1008 = 2^4 \cdot 3^2 \cdot 7$. This contradiction proves $\bar{Q} = 1$.

As $N_{\overline{M}}(\bar{E}_2) = \bar{E}_2 \cdot \bar{A}_c \cong \mathrm{Hol}(E(2^4))$, we are in a position to apply the main result of [11]. So $\overline{M}/O(\overline{M})$ is isomorphic either to $GL_5(2)$ or the Mathieu group M_{24}.

As $\mid O(\overline{M})|$ divides $3^4 \cdot 5 \cdot 7^2 \cdot 11 \cdot 17 \cdot 23 \cdot 31^2 \cdot 73 \cdot 89 \cdot 127$, we easily see that $\overline{M}/O(\overline{M})$ induces only the trivial automorphism on $O(\overline{M})$. So $[O(\overline{M}), \bar{E}_2] = 1$; this implies $O(\overline{M}) = 1$.

Now suppose $\overline{M} \cong GL_5(2)$. Then there exists an element q in M with $o(\bar{q}) = 31$ and $\mid C_U(q)| \in \{2, 2^6\}$. Hence, there exists $u \in C_U(q)^{\#}$ with $\mid \overline{M}: C_{\overline{M}}(u)| \leqslant 2^{11}$ and $31 \mid \mid C_{\overline{M}}(u)|$. By the structure of $GL_5(2)$ we get $C_{\overline{M}}(u) = \overline{M}$. But we also have $\mid \mathrm{ccl}_{\overline{M}}(x)| \neq 1$ for each $x \in U^{\#}$ by the results in (2.2). This contradiction implies $\overline{M} \cong M_{24}$. Q.E.D.

(2.6) The following statements are equivalent:

(1) $A_c \cap C(s) = A_c$.

(2) $N_G(U) = C_G(U)\, E_2 A_c$.

(3) G contains a subgroup with index 2.

Proof. Let $M = N_G(U)$ and $\overline{M} = M/C_G(U)$.

(1) $\Rightarrow$ (2). Suppose $\overline{M} > \bar{E}_2 \bar{A}_c$ and $[A_c, s] = 1$. Clearly, $\overline{M} \cong M_{24}$. So, by results of [14], either $(M!\ U) = (759) + (1288)$ or $(M!\ U) = (1771) +$

(*276*). Making use of (2.2) (c) and inspecting the structure of M_{24} we get $(M!\, U) = 759v_1 + 1288v_2 s$ with $C_{\overline{M}}(v_1) \cong \mathrm{Hol}(E(2^4))$. Clearly, $O_2(C_{\overline{M}}(v_1)) = \bar{E}_i$ and $C_{\overline{M}}(v_1) = N_{\overline{M}}(\bar{E}_i)$ for some $i \in \{1, 2\}$. Note that there exists an element q in A_c such that $q \cdot O(A_2) = q_0 \cdot O(A_2)$ and $\bar{q} \in N_{\overline{M}}(\bar{E}_1) \cap N_{\overline{M}}(\bar{E}_2)$. Therefore $v_1 = v_1{}^q = v_1^{q_0}$; but this is absurd.

(2) $\Rightarrow$ (1). Suppose $\overline{M} = \bar{E}_2 \bar{A}_c$ and $[A_c, s] \neq 1$. Then $r_1{}^s = v_2{}^y r_1$, $r_0^{sr_2} = v_1^{q_0 q_1} \in \mathrm{ccl}_L(v_1)$, and $r_0^{q_1{}^6 s q_1} = v_2^{y q_1} r_0 \in \mathrm{ccl}_{N_{VL}(E)}(v_2^{y k_3} r_0)$. Since $N_G(U)$ controls G-fusion in U, we get $v_2^{y k_3} r_0 \in \mathrm{ccl}_{E_2 A_c}(v_1)$ which is obviously impossible.

(3) $\Rightarrow$ (2). Suppose $\overline{M} \cong M_{24}$. As $23 \nmid |GL_{10}(2)|$, M acts irreducible on U. Thus G contains no subgroups with index two.

(2) $\Rightarrow$ (3). Assume now that G contains no subgroups with index two. Thus, there exists an element $g \in G$ with $s^g \in VT$. If $s^g \in V$, then $s \in \mathrm{ccl}_G(v_i)$ for some $i \in \{1, 2\}$. Henceforth we assume that $s^g \in VT - V$. So we may assume that $s^g \in Vr_0$ or $s^g \in Vr_1 r_2$. Note that $(H_1!\, C_V(r_1 r_2)) = [V, r_1 r_2] + 12 v_1^{q_0} + 12 v_2{}^y + 24 v_2^{y q_1 t_1}$ and $\{v_1^{q_0} r_1 r_2, v_2^{y q_1 t_1}\} \subseteq \mathrm{ccl}_{N_{VL}(E)}(v_2{}^y r_1 r_2)$. Since $(r_1 r_2)^s \in v_1 v_1^{k_2} r_1 \cdot \langle v_2{}^y \rangle$, $v_2^{ys} = v_2{}^y$, and $r_1^{q_1} = r_0$, we may assume that $s^g \in Vr_0$. Hence, $s^g \in C_V(r_0)\, r_0 = C_V(E_2) \langle v_1^{q_0 q_1} \rangle r_0$. As $r_0{}^s = v_1 v_1^{q_0 q_1}$, we have either $s^{gs} \in V$ or $s^{gs} \in C_V(E_2)\, v_1^{q_0 q_1} r_0 \subseteq W^* \leqslant U$.

Recall $\langle v_1, v_2 \rangle \leqslant W^*$. We have proved that in any case there exists an element g' of G such that $s^{g'} \in W^*$. Since $N_G(U)$ controls the G-fusion in U and $E_2 A_c \leqslant N_G(W^*)$, we finally get $N_G(U) > C_G(U)\, E_2 A_c$. Q.E.D.

(2.7) Let $N_G(U)/C_G(U) \cong M_{24}$; then the following hold:

(a) $(N_G(U)!\, U) = 1771 v_1 + 276 v_2$ with

$$(E_2 A_c!\, \mathrm{ccl}_{N_G(U)}(v_1)) = 35 v_1 + 840 v_2^{y k_3} r_0 + 896 v_1 s$$

and

$$(E_2 A_c!\, \mathrm{ccl}_{N_G(U)}(v_2)) = 28 v_2 + 120 v_1^{q_0 q_1} r_0 + 128 s.$$

(b) The group G contains exactly two conjugacy classes of involutions and v_1 and v_2 are representatives of these classes.

Proof. (a) By results of [14], either $(N_G(U)!\, U) = (759) + (1288)$ or $(N_G(U)!\, U) = (1771) + (276)$. Making use of (2.2) (c) we easily deduce the required results.

(b) To prove the statement we only may consider the involutions in VT, because G contains no subgroups with index two and $S = VT\langle s \rangle \in \mathrm{Syl}_2(G)$.

Thus, let x be an involution of VT. Using the same arguments as in the last part of the proof of (2.6) we see that there exists an element g of G such that $x^g \in W^*$. Since $W^* < U$ and $N_G(U)$ controls G-fusion in U, x^g is conjugate either to v_1 or to v_2 and $v_1 \notin \mathrm{ccl}_G(v_2)$. Q.E.D.

THEOREM 2. *Let G be a finite group satisfying* (A1) *and* (A2); *furthermore, suppose G contains a subgroup G_1 with index two.*

Then there exists a normal series $1 \trianglelefteq O(G) \triangleleft G_0 \trianglelefteq G_1 \triangleleft G$ *of subgroups of* G *satisfying the following:*

(1) $VT \in \mathrm{Syl}_2(G_0) = \mathrm{Syl}_2(G_1)$ *and*

$$G_1/G_0 \cong N_{G_1}(VT)/N_{G_0}(VT) \cong O(C_{G_1}(VT))/O(C_{G_0}(VT)).$$

(2) $G_0/O(G)$ *is a non-Abelian simple group and*

$$N_{G_0}(V)/C_{G_0}(V) \cong GL_5(2).$$

(3) G_0 *contains at least four different conjugacy classes of involutions which are already* G*-classes.*

Proof. Let G_1 be a subgroup of G with $|G: G_1| = 2$. Obviously, $VT \in \mathrm{Syl}_2(G_1)$ and $N_{G_1}(V)/C_{G_1}(V) \cong GL_5(2)$. Moreover $O(G_1) = O(G)$.

By (1.16) there exists a normal subgroup G_0 of G_1 such that $VT \in \mathrm{Syl}_2(G_0)$, $O(G_0) = O(G)$, and $G_1/G_0 \cong N_{G_1}(VT)/N_{G_0}(VT) \cong O(C_{G_1}(VT))/O(C_{G_0}(VT))$; furthermore, $G_0/O(G)$ is a non-Abelian simple group.

Making use of the results proved so far we easily obtain all the claimed statements. Q.E.D.

3. Groups Related to J_4

In this section the group G is supposed to satisfy the additional assumption

(A3) The group G contains no subgroups with index two.

Hence, $N_G(U) = U \cdot M$ with $O(M) = O(N_G(U)) = O(C_G(U))$ and $M/O(M) \cong M_{24}$, because $S = U \cdot T \in \mathrm{Syl}_2(N_G(U))$.

Moreover, we use the following notation:

$$N = N_{VL}(E) = O(C_L(E))\, EV_1K_1 \langle a_1, t_1\rangle,\ v = v_1v_2,$$

and $P = \langle v, v^{q_1^6}d_2, v^{q_1^5}t_2, s\rangle$. Thus $P \cong E(2^4)$ and $UE = U \cdot R_1 = E \cdot P$.

(3.1) (a) We have $(N_G(E)!\,E) = (\langle N, s\rangle !\,E) = 1v_1 + 33 \cdot 2 \cdot 3 \cdot 7v_1^{q_0} + 66 \cdot 2 \cdot 3 \cdot 7{v_2}^y + 2^6 \cdot 3^2 \cdot 7v_1^{k_2}r_0$, where ${v_1}^2 = (v_1^{q_0})^2 = ({v_2}^y)^2 = 1$ and $(v_1^{k_2}r_0)^2 = v_1$.

(b) $C_G(E) = \langle v_1\rangle \times O(C_G(E))$ and $N_G(E)/(EC_G(E)) \cong \hat{3}\,\mathrm{Aut}(M_{22})$.

Proof. (a) Clearly, $s \in N_G(E)$ and ${r_0}^s = v_1v_1^{q_0q_1}$, $({v_2}^yr_0)^s = v_2^{yk_3}$, $(v_1^{q_0q_1}r_0)^{q_1^6sq_1} = v_1^{q_2q_1}r_0 = v_1^{q_0q_2}r_0$, $(v_2^{yq_1}r_0)^{q_1^6sq_1} = r_0$, $(r_1r_2)^{sq_1} = v_1v_1^{k_2q_1}v_2^{yq_1}r_0 = v_1^{q_2}v_2^{yq_1}r_0 \in \mathrm{ccl}_T(v_2^{yq_1}r_0)$, and $({v_2}^yr_1r_2)^{sq_1} = v_1v_1^{k_2q_1}r_0 = v_1^{q_2}r_0$.

Furthermore, $(v_1^{q_0q_1}r_1r_2)^s = {v_2}^yv_1^{k_2}r_0r_1 \in \mathrm{ccl}_N(v_1^{k_2}r_0)$ and $(v_2^{yq_1^3}r_1r_2)^s = {v_2}^yv_2^{yq_1^3}r_1r_2 \in v_2^{yq^2_1}r_1r_2\langle v_1\rangle \subseteq \mathrm{ccl}_N(v_1^{q_0q_1}r_1r_2)$.

Making use of (1.12) and (2.7) we now easily verify the claimed results.

(b) Note that $C_S(E) = C_{VT}(E) = C_V(R_1) = \langle v_1\rangle = Z(E)$; since $E \triangleleft S$, $Z(E)$ is an S_2-subgroup of $C_G(E)$. Therefore $C_G(E) = O(C_G(E)) \times Z(E)$.

Note that $S = E \cdot P \cdot (K_{12} \times \langle t_1 \rangle)$. Since $\langle a_1, a_3, b_1t_1, b_2t_1, t_1d_{2+} \rangle \leqslant A \leqslant C_{A_c}(s)$ and $\langle [a_1, d_{2+}], [a_3, t_1] \rangle \leqslant O(A_c)$, we get $1 \neq [t_1, s] = [d_{2+}, s] \in C_V(E_2, t_1, d_{2+}, a_1, a_3) = \langle v_1, v_2 \rangle$. As $s \in \mathrm{ccl}_M(v_2)$ and $v_1s \in \mathrm{ccl}_M(v_1)$ as well as $(v_1v_2s)^{b_1t_1} = v_1s$, we see that $[d_{2+}, s] = [t_1, s] = v_2 = v_1v$. So the following relations hold:

$$[t_1, v] = [t_1, v^{{a_1}^6}d_2] = [t_1, v^{{a_1}^5}t_2] = 1, \qquad [t_1, s] = v_1v;$$

$$[t_2, v] = [t_2, v^{{a_1}^5}t_2] = 1, \qquad [t_2, v^{{a_1}^6}d_2] = v, \qquad [t_2, s] = v^{{a_1}^5}t_2;$$

$$[d_2, v] = [d_2, v^{{a_1}^6}d_2] = 1, \qquad [d_2, v^{{a_1}^5}t_2] = v, \qquad [d_2, s] = v^{{a_1}^6}d_2;$$

$$[d_{2+}, v] = [d_{2+}, v^{{a_1}^6}d_2] = 1, \qquad [d_{2+}, v^{{a_1}^5}t_2] = v^{{a_1}^2}d_2, \qquad [d_{2+}, s] = v_1v.$$

Note that a_1 has no fixed points on $E - Z(E)$; hence $F_0 = \langle C_N(a_1 \bmod O(N)), s \rangle = \langle O(N), v_1, a_1, V_1K_1, s \rangle$ and $F = \langle N_N(\langle a_1 \rangle \bmod O(N)), s \rangle = F_0 \cdot \langle t_1 \rangle$. Moreover, $\langle N, s \rangle = EF$ with $E \cap F = E \cap F_0 = Z(E)$ and $2^8 \cdot 3^3 \cdot 7 \cdot 11 \mid |F/C_F(E)|$.

Put $D = N_G(E)$, $\bar{D} = D/C_G(E)$, $\tilde{D} = D/(EC_G(E))$, and $F_2 = S \cap F = \langle v_1, V_1K_{12}, s, t_1 \rangle$; then $\bar{S} = \bar{E} \cdot \bar{F}_2 \in \mathrm{Syl}_2(\bar{D})$ and so $\bar{D} = \bar{E} \cdot \bar{C}$ for some subgroup C of G with $C \cap E \leqslant C_G(E)$. Since $|\bar{D}|$ divides $|\mathrm{Aut}(E)|$, we get $2^8 \cdot 3^3 \cdot 7 \cdot 11 \mid |\bar{C}| \mid 2^8 \cdot |\mathrm{Aut}(E)|_{2'} = 2^8 \cdot 3^8 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 17 \cdot 31$.

Clearly, $(\bar{D} ! \bar{E}) = 3^2 \cdot 7 \cdot 11\overline{v_1^{q_0}} + 2 \cdot 3^2 \cdot 7 \cdot 11\overline{{v_2}^y} + 2^5 \cdot 3^2 \cdot 7\overline{v_1^{k_2}r_0}$ and $3^2 \cdot 7 \cdot 11 = 31 \cdot 22 + 11 = 17 \cdot 40 + 13$, $2 \cdot 3^2 \cdot 7 \cdot 11 = 31 \cdot 44 + 22 = 17 \cdot 81 + 9$, and $2^5 \cdot 3^2 \cdot 7 = 31 \cdot 65 + 1 = 17 \cdot 118 + 10$.

Now suppose that there exists an element x of D with $o(\bar{x}) = 17$. On the one hand we certainly have $|C_{\bar{E}}(\bar{x})| \geqslant 1 + 13 + 9 + 10 = 33$, but on the other hand, by a theorem of Maschke, $|C_{\bar{E}}(\bar{x})| = 2^4$. So $17 \nmid |\bar{D}|$.

Let $x \in D$ with $o(\bar{x}) = 31$ and let X_1 and X_2 be subgroups of E containing $Z(E)$ such that $\bar{X}_1 = C_{\bar{E}}(\bar{x})$ and $\bar{X}_2 = [\bar{E}, \bar{x}]$. Then $X_1 \simeq E(2^7)$ and $X_2 = [X_2, x] \times Z(E) \simeq E(2^6)$. Hence there exist $x_1 \in X_1$ and $x_2 \in [X_2, x]$ with $[x_1, x_2] = v_1$. So $v_1 = [x_1, x_2]^x = [x_1, {x_2}^x]$ and $x_2{x_2}^x = x_2^{x^n}$ for some $n \in \{2, 3, \ldots, 30\}$. Thus, $1 = [x_1, x_2{x_2}^x] = [x_1, x_2]^{x^n} = v_1$, which is absurd. Therefore $31 \nmid |\bar{D}|$.

Put $Q = O(M)$ and $M_1 = C_M(v_1)$; then $M_1/Q \simeq E(2^6) \cdot (\hat{3}\Sigma_6)$. Since $S = U \cdot T \in \mathrm{Syl}_2(UM_1)$ and $a_3 \in UM_1$ with ${a_3}^2 \colon d_2 \to t_2$, we get $O_{2'2}(UM_1) = Q \cdot UR_1$ and $UM_1 = QUR_1Y$ with $Q \lhd Y$ and $Y/Q \simeq \hat{3}\Sigma_6$. So the Frattini argument yields $UM_1 = QN_{UM_1}(UR_1) = QN_{UM_1}(E)$. Moreover, we have $O(N_{UM_1}(E)) = N_Q(E) = C_Q(E)$ and $C_{UM_1}(E) = Z(E) \times C_Q(E)$ as well as $E(2^4) \simeq \tilde{P} = C_{\tilde{S}}(\tilde{P}) \lhd \tilde{S} = \tilde{P} \cdot (\widetilde{K_{12} \times \langle t_1 \rangle})$. So $\bar{D} \gtrsim N_{UM_1}(E)/C_{UM_1}(E) \simeq E(2^{12}) \cdot E(2^4) \cdot (\hat{3}\Sigma_6)$ and $N_{\tilde{D}}(\tilde{P}) = \widetilde{N_{UM_1}(E)} \simeq E(2^4) \cdot (\hat{3}\Sigma_6)$.

We have proved so far that $\bar{D} = \bar{E} \cdot \bar{C}$ with $2^8 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \mid |\bar{C}| = |\tilde{D}| \mid 2^8 \cdot 3^8 \cdot 5^2 \cdot 7^2 \cdot 11$ and $\tilde{D} > N_{\tilde{D}}(\tilde{P}) \simeq E(2^4) \cdot (\hat{3}\Sigma_6)$ as well as $\tilde{D} > \widetilde{N_{VL}(E)} = \tilde{V}_1\tilde{K}_1 \times \langle \tilde{a}_1, \tilde{t}_1 \rangle \simeq E(2^3) \cdot GL_3(2) \times \Sigma_3$.

Now let x be an involution of $\tilde{P}\tilde{K}_{12} - \tilde{P}$. Note that $a_3{}^2\colon v \to v,\ s \to s,$ $v^{q_1{}^6} \to v^{q_1{}^5}$, $d_2 \to t_2$ and hence $\tilde{a}_3 \in N_{\tilde{D}}(\tilde{P})$. As K_{12} is dihedral of order 8 with $[d_{2+}, t_2] = d_2$, the involution x is conjugate to an element of $\tilde{P}\tilde{d}_2$ or $\tilde{P}\tilde{d}_{2+}$. Since $C_{\tilde{P}}(\tilde{d}_2) = C_{\tilde{P}}(\tilde{d}_{2+}) \cong E(2^2)$, x is conjugate to $\tilde{d}_2$ or $\tilde{d}_{2+}$.

So any involution of $\tilde{P}\tilde{K}_{12}$ is conjugate under $N_{\tilde{D}}(\tilde{P})$ to an element of $\{\tilde{v}, \tilde{d}_2, \tilde{d}_{2+}\}$. As $C_E(v) = V_0\langle r_0, r_2\rangle$, $C_E(d_2) = C_{V_0}(E_2) \cdot (R_1 \cap R_2)$, and $C_E(d_{2+}) = C_{V_0}(E_2)^{a_2{}^2} \cdot (R_1 \cap R_2)$, we easily see that $C_{\bar{E}}(x) \cong E(2^8)$ for any involution x of $\tilde{P}\tilde{K}_{12}$.

Note that $\tilde{t}_1$ is an involution of $\tilde{S} - \tilde{P}\tilde{K}_{12}$ with $C_{\bar{E}}(\tilde{t}_1) = \overline{C_V(E_1)}\,\langle \bar{r}_0, \bar{r}_1, \bar{d}_1\rangle \cong E(2^6)$. Since $\tilde{S} = \tilde{P}\tilde{K}_{12} \cdot \langle \tilde{t}_1\rangle \in \mathrm{Syl}_2(\tilde{D})$, application of a theorem of Thompson yields the existence of a subgroup D_0 of D with $|\, D\colon D_0 \,| = |\, \tilde{D}\colon \tilde{D}_0 \,| = 2$.

Note that $C_{\tilde{D}}(\tilde{P}) = \tilde{P} \times O(C_{\tilde{D}}(\tilde{P}))$ and that Σ_6 is a maximal subgroup of Alt_8. So $\tilde{P} < O_{22'}(N_{\tilde{D}}(\tilde{P})) = O_{2'2}(N_{\tilde{D}}(\tilde{P})) = C_{\tilde{D}}(\tilde{P}) \cong E(2^4) \times Z_3$ and $N_{\tilde{D}}(\tilde{P})/C_{\tilde{D}}(\tilde{P}) \cong \Sigma_6$. Hence, $\tilde{P}\tilde{K}_{12} \in \mathrm{Syl}_2(\tilde{D}_0)$ and $C_{\tilde{D}_0}(\tilde{P}) = C_{\tilde{D}}(\tilde{P})$ as well as $N_{\tilde{D}_0}(\tilde{P})/C_{\tilde{D}_0}(\tilde{P}) \cong \mathrm{Alt}_6$. Clearly, $\tilde{D}_0$ contains no subgroups the index of which is a power of 2.

Put $\tilde{\tilde{D}} = \tilde{D}_0/O(\tilde{D}_0)$; by results of [10], $\tilde{\tilde{D}}_0$ has sectional 2-rank four. Moreover, $\tilde{\tilde{D}}_0$ is not solvable and $Z(\tilde{\tilde{D}}_0) = O(\tilde{\tilde{D}}_0) = 1$. We are in a position to make use of the main results of [8]:

Since $\tilde{\tilde{F}}_0 = \langle \tilde{\tilde{a}}_1, \tilde{\tilde{V}}_1\tilde{\tilde{K}}_1, \tilde{\tilde{s}}\rangle \leqslant \tilde{\tilde{D}}_0$ and $\tilde{\tilde{P}} < (\tilde{\tilde{D}}_0)'$ with $N_{(\tilde{\tilde{D}}_0)'}(\tilde{\tilde{P}})/C_{(\tilde{\tilde{D}}_0)'}(\tilde{\tilde{P}}) \cong \mathrm{Alt}_6$, $(\tilde{\tilde{D}}_0)'$ is a simple group of sectional 2-rank 4 with $\mathrm{SCN}_3(2) \neq \varnothing$ and $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \mid |(\tilde{\tilde{D}}_0)'|$; hence, $(\tilde{\tilde{D}}_0)'$ is isomorphic either to Alt_{11} or to the Mathieu group M_{22}.

As $\tilde{\tilde{K}}_1 \leqslant (\tilde{\tilde{D}}_0)'$, $\tilde{\tilde{P}}$ is not strongly closed in $\tilde{\tilde{P}}\tilde{\tilde{K}}_{12}$ with respect to $(\tilde{\tilde{D}}_0)'$. Since $(\tilde{\tilde{D}}_0)'$ contains a section isomorphic to Alt_6 which acts on $\tilde{\tilde{P}}$ and since Alt_6 has only one class of involutions, we easily see that $(\tilde{\tilde{D}}_0)'$ has exactly one class of involutions. So $(\tilde{\tilde{D}}_0)' = \tilde{\tilde{D}}_0 \cong M_{22}$ and $|\, \tilde{\tilde{D}}_0 \,| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$.

Clearly, $O(\tilde{D}) = O(\tilde{D}_0)$ and $C_{\tilde{D}}(\tilde{P}) = C_{\tilde{D}_0}(\tilde{P})$. Since there is an involution in $\tilde{D} - \tilde{D}_0$ which induces an outer automorphism on $N_{\tilde{D}_0}(\tilde{P})/C_{\tilde{D}}(\tilde{P})$, we have $\tilde{D}/O(\tilde{D}) \cong \mathrm{Aut}(M_{22})$ and $3 \mid |\, O(\tilde{D})| \mid 3^6 \cdot 5 \cdot 7$.

Note that $\tilde{D}_0/O(\tilde{D})$ induces only the trivial automorphism on $O(\tilde{D})$. Thus, elements of $O(\tilde{D})^{\#}$ have no fixed points in $\bar{E}^{\#}$. So $C_{\tilde{D}_0}(x) \cong O(\tilde{D}) \cdot C_{\tilde{D}_0}(x)/O(\tilde{D}) \lesssim M_{22}$ and $|\, \mathrm{ccl}_{\tilde{D}_0}(x)| = |\, O(\tilde{D})| \cdot |\, M_{22} \,|/|\, C_{\tilde{D}_0}(x)|$ for any $x \in \bar{E}^{\#}$.

We have $3^2 \cdot 7 \cdot 11 = |\, \mathrm{ccl}_{\tilde{D}}(\overline{v_1^{q_0}})| = |\, \mathrm{ccl}_{\tilde{D}_0}(\overline{v_1^{q_0}})|$; since three divides $|\, O(\tilde{D})|$ and since $E(2^4) \cdot \mathrm{Alt}_6$ contains no subgroups with index 3, $C_{\tilde{D}_0}(\overline{v_1^{q_0}})$ is isomorphic either to $E(2^4) \cdot \Sigma_5$ or to $E(2^4) \cdot \mathrm{Alt}_6$. So $|\, O(\tilde{D})|$ divides 3^2.

Furthermore, we have $|\, \mathrm{ccl}_{\tilde{D}_0}(\overline{v_1^{k_2}r_0})| \in \{2^5 \cdot 3^2 \cdot 7, 2^4 \cdot 3^2 \cdot 7\}$. Thus $2^2 \cdot 3 \cdot 5 \cdot 11 \mid |\, C_{\tilde{D}_0}(\overline{v_1^{k_2}r_0})| \mid 2^3 \cdot 3^2 \cdot 5 \cdot 11$. Inspecting the structure of M_{22} we get $C_{\tilde{D}_0}(\overline{v_1^{k_2}r_0}) \cong L_2(11)$ which has order $2^2 \cdot 3 \cdot 5 \cdot 11$. Therefore $O(\tilde{D}) \cong Z_3$.

As $N_{\tilde{D}}(\tilde{P})/\tilde{P} \cong \hat{3}\Sigma_6$, we finally conclude that $\tilde{D}_0 \cong \hat{3}M_{22}$ and $\tilde{D} \cong \hat{3}\mathrm{Aut}(M_{22})$.

Q.E.D.

(3.2) $C_G(v_1) = O(C_G(v_1))\, N_G(E)$; in particular, $C_G(v_1)$ is 2-constrained.

Proof. Let $C = C_G(v_1)$, $\bar{C} = C/\langle v_1\rangle$, $D = N_G(E)$, $\tilde{D} = D/(EC_G(E))$, and $S_0 = UR_1 \cdot K_{12} = E \cdot P \cdot K_{12}$.

We already know that D contains a subgroup D_0 such that $D = D_0 \cdot \langle t_1\rangle$, $\tilde{D}_0 \cong \hat{3}M_{22}$, and $\tilde{S}_0 = \tilde{P}\tilde{K}_{12} = \tilde{V}_1\tilde{K}_{12} \cdot \langle \tilde{s}\rangle \in \mathrm{Syl}_2(\tilde{D}_0)$.

Now put $X_0 = \langle v, v^{q_1^6}, d_2, d_{2+}\rangle$ and $X_1 = X_0 \times \langle t_1\rangle$; then $\tilde{X}_1$ is the only elementary Abelian subgroup of order 2^5 in $\tilde{S}$, and $\tilde{P}$ and $\tilde{X}_0$ are the only elementary Abelian subgroups of order 2^4 in $\tilde{S}_0$. It is well known that $\tilde{D}$ has exactly three conjugacy classes of involutions and that two classes of them are contained in $\tilde{D} - \tilde{D}_0$; representatives of these two classes may be found in $\tilde{X}_1 - \tilde{X}_0$. Moreover, $N_{\tilde{D}_0}(\tilde{X}_0)$ acts on $\tilde{X}_1 - \tilde{X}_0$ inducing orbits of lengths 10 and 6 on this set.

We have $a_2\colon t_1 \to t_1$, $d_2 \to d_2 d_{2+} \to d_{2+}$, $v \to v^{q_1^2} \to v^{q_1^6}$ and $s\colon d_2 \to v^{q_1^6}$, $d_{2+} \to v_1 v d_{2+}$, $t_1 \to v_1 v t_1$; hence $\langle \tilde{a}_2, \tilde{s}\rangle \leqslant N_{\tilde{D}_0}(\tilde{X}_0)$. So $\tilde{t}_1$ and $\tilde{v}\tilde{d}_{2+}\tilde{t}_1$ are representatives of the two conjugacy classes of involutions in $\tilde{D} - \tilde{D}_0$.

Now calculate $\{x \in E \mid [x, v] \in \langle v_1\rangle\} = C_E(v) = V_0\langle r_0, r_2\rangle$, $\{x \in E \mid [x, vd_{2+}t_1] \in \langle v_1\rangle\} = C_E(vd_{2+}t_1)\,\langle v_2^{ya_1} r_3 d_1\rangle = \langle v_1, v_1^{q_0^2}, v_1^{q_0 q_1}, v_2^{y}, r_0, v_1^{q_0} r_1, v_2^{ya_1} r_3 d_1\rangle$, and $\{x \in E \mid [x, t_1] \in \langle v_1\rangle\} = C_E(t_1) = C_V(E_1)\,\langle r_0, r_1, d_1\rangle$. So we conclude that

$$C_{\bar{E}}(x_1) \cong E(2^8) \qquad \text{for any involution } x_1 \in \bar{D}_0 - \bar{E}$$

and

$$C_{\bar{E}}(x_2) \cong E(2^6) \qquad \text{for any involution } x_2 \in \bar{D} - \bar{D}_0 .$$

As E is the only subgroup isomorphic to $Ex(2^{13})$ in the S_2-subgroup S of C, $\bar{E}$ is weakly closed in $\bar{S}$ w.r.t. $\bar{C}$.

In what follows we assume by way of contradiction that $\bar{E}$ is not strongly closed in $\bar{S}$ w.r.t. $\bar{C}$.

Let $\beta = \{\bar{B} \mid \bar{B} \leqslant \bar{S}, \bar{B} \nleqslant \bar{E}, \exists\, g \in \bar{C}\colon \bar{B}^g \leqslant \bar{E}\}$ and let r be the maximal element of $\{m(\bar{B}/C_{\bar{B}}(\bar{E})) \mid \bar{B} \in \beta\}$. We are in a position to make use of results of [6, Chap. 9]. So the following statements hold for any $\bar{B}$ contained in β:

(1) $C_{\bar{B}}(\bar{E}) = \bar{B} \cap \bar{E}$.

(2) There exists an element g of $\bar{C}$ such that $\bar{B}^g \leqslant \bar{E}$ and $N_{\bar{S}}(\bar{B})^g \leqslant \bar{S}$.

(3) There exists an element $\bar{B}_0$ of β such that $m(\bar{B}_0) + r \geqslant m(\bar{E}) = 12$.

(4) If x is an involution of $\bar{S}$ with $\mathrm{ccl}_{\bar{C}}(x) \cap \bar{E} \neq \varnothing$, then $m([\bar{E}, x]) \leqslant r$.

Since $r \leqslant m(\bar{S}/\bar{E}) = 5$, $[\bar{E}, x] = C_{\bar{E}}(x) \cong E(2^6)$ for any involution $x \in \bar{S} - \bar{S}_0$, and $m([\bar{E}, x]) = m(\bar{S}_0/\bar{E}) = 4$ for any involution $x \in \bar{S}_0 - \bar{E}$, we have the following:

(5) If $\bar{B} \in \beta$, then $\bar{B} \leqslant \bar{S}_0$.

(6) $r = 4$.

Let $\bar{B}$ be an element of maximal order of β; hence, by (3) and (6), 2^8 divides $|\bar{B}|$. Moreover, let x be an element of $\bar{B} - \bar{E}$ and set $\bar{B}_1 = (\bar{B} \cap \bar{E}) \cdot \langle x\rangle$ and $\bar{B}_2 = N_{\bar{E}}(\bar{B}_1)$.

Clearly, $[\bar{E}, x](\bar{B} \cap \bar{E}) \leqslant C_{\bar{E}}(x) \cong E(2^8)$. Therefore $|\bar{B}_2| = |C_{\bar{E}}(x)| \cdot |[\bar{E}, x] \cap \bar{B} \cap \bar{E}| \geqslant |[\bar{E}, x]| \cdot |\bar{B} \cap \bar{E}| = 2^4 \cdot |\bar{B} \cap \bar{E}|$. Furthermore, there exists an element g in $\bar{C}$ such that $(\bar{B}_1)^g \leqslant \bar{E}$ and $(\bar{B}_2)^g \leqslant \bar{S}$. If $|\bar{B}_2| > |\bar{B}|$, we got $(\bar{B}_2)^g \leqslant \bar{E}$, because $\bar{B}$ is an element of maximal order of β; hence $\bar{B}_1\bar{B}_2 \in \beta$ and thus $|\bar{B}_2| > |\bar{B}| \geqslant |\bar{B}_1\bar{B}_2|$ which is absurd. So $|\bar{B}| \geqslant |\bar{B}_2| \geqslant 2^4 \cdot |\bar{B} \cap \bar{E}|$. As $m(\bar{B}/\bar{B} \cap \bar{E}) \leqslant 2^r = 2^4$, we get

(7) $|\bar{B}| = |\bar{B}_2| = 2^4 \cdot |\bar{B} \cap \bar{E}|$.

Therefore $E(2^4) \cong \bar{B}/\bar{B} \cap \bar{E} \cong \bar{E}\bar{B}/\bar{E} \leqslant \bar{S}_0/\bar{E} \cong \tilde{S}_0$. We know that $\tilde{P}$ and $\tilde{X}_0$ are the only subgroups of $\tilde{S}_0$ which are isomorphic to $E(2^4)$. So either $\bar{E}\bar{B} = \bar{E}\bar{P} = \bar{E}\bar{U}$ or $\bar{E}\bar{B} = \bar{E}\bar{X}_0$.

Suppose $\bar{E}\bar{B} = \bar{E}\bar{U}$. Then $\bar{B} \cap \bar{E} \leqslant Z(\bar{E}\bar{U}) = \bar{E} \cap \bar{U}$ and $[\bar{E}, x] \leqslant (\bar{E}\bar{U})' \leqslant \bar{E} \cap \bar{U} \cong E(2^6)$; so $|\bar{B}| = |\bar{B}_2| = |C_{\bar{E}}(x)| \cdot |[\bar{E}, x] \cap \bar{B} \cap \bar{E}| \geqslant 2^2 \cdot |[\bar{E}, x]| \cdot |\bar{B} \cap \bar{E}| = 2^6 \cdot |\bar{B} \cap \bar{E}|$, but this contradicts (7).

So $\bar{E}\bar{B} = \bar{E}\bar{X}_0$ and $\bar{B} \cap \bar{E} \leqslant Z(\bar{E}\bar{X}_0) = C_{\bar{E}}(\bar{X}_0)$. As $v = v_1v_2 \in \mathrm{ccl}_G(v_1)$ and $v_2 \notin \mathrm{ccl}_G(v_1)$, we have $C_{\bar{E}}(\bar{X}_0) \leqslant \overline{C_E(v, v^{q_1^6})} = \bar{V}_0$. Note that $[V_0, d_2] = [V_0, d_{2+}] = \langle v_1^{q_0^2}, v_1^{q_2}\rangle$; so $C_{\bar{E}}(\bar{X}_0) = \overline{C_{V_0}(d_2, d_{2+})} = \overline{C_V(R_2)} \cong E(2^2)$. Hence $|\bar{B}| = 2^4 \cdot |\bar{B} \cap \bar{E}| \leqslant 2^6$. But this contradicts the fact that 2^8 divides $|\bar{B}|$.

We have proved that $\bar{E}$ is strongly closed in $\bar{S}$ w.r.t. $\bar{C}$. So $\bar{C}$ is a $S^*(\bar{E})$-group in the sense of [6].

Let $H = E^C$; then $\bar{H} = \bar{E}^{\bar{C}} \trianglelefteq \bar{C}$ and $O(\bar{H}) \cdot \bar{E} = O_{2'2}(\bar{H})\,\Omega_1(\bar{X})$ for any S_2-subgroup $\bar{X}$ of $\bar{H}$ containing $\bar{E}$. Since $\bar{S}$ splits over $\bar{E}$, the group $\bar{E}$ is an S_2-subgroup of $\bar{H}$.

We have $\bar{E} \leqslant C_{\bar{H}}(\bar{E}) \leqslant N_{\bar{H}}(\bar{E}) = \bar{H} \cap \bar{D} \trianglelefteq \bar{D}$ and $\bar{D}/O(\bar{D}) \cong E(2^{12}) \cdot \hat{3}\mathrm{Aut}(M_{22})$; thus $N_{\bar{H}}(\bar{E})/C_{\bar{H}}(\bar{E}) \leqslant Z_3$. Moreover, as $\bar{C}$ is an $S^*(\bar{E})$-group, $\bar{H}/O(\bar{H})$ is isomorphic to a central product of an elementary Abelian 2-group and some quasi-simple groups of known type.

Now suppose that $N_{\bar{H}}(\bar{E}) \neq C_{\bar{H}}(\bar{E})$. Then $\bar{H} > O(\bar{H}) \cdot \bar{E}$ and $\bar{D}/(\bar{H} \cap \bar{D}) \cong \mathrm{Aut}(M_{22})$ acts on both, $\bar{H}$ and $\bar{E}$; so we conclude that $\bar{H}/O(\bar{H})$ is a simple group isomorphic to $L_2(2^n)$, $Sz(2^{2n+1})$, or $U_3(2^n)$ for some suitable integer n. As $\bar{E} \in \mathrm{Syl}_2(\bar{H})$, the group $\bar{H}/O(\bar{H})$ is isomorphic either to $L_2(2^{12})$ or to $U_3(2^4)$. But this is absurd, because $N_{\bar{H}}(\bar{E})/C_{\bar{H}}(\bar{E}) \cong Z_3$ and $\mathrm{Aut}(M_{22}) \leqslant \mathrm{Aut}(\bar{H}/O(\bar{H}))$.

We have proved that $N_{\bar{H}}(\bar{E}) = C_{\bar{H}}(\bar{E})$. Application of a theorem of Burnside yields $\bar{H} = O(\bar{H}) \cdot \bar{E}$. Making use of the Frattini argument we finally get $C = O(C)\,N_G(E)$.

Since $E \in \mathrm{Syl}_2(O_{2'2}(C))$ and $C_G(E) = O(C_G(E)) \times Z(E) = O(N_G(E)) \times Z(E) \leqslant O_{2'2}(C)$, the group C is 2-constrained. Q.E.D.

Notation. In what follows let $C_1 = C_G(v_1, v_2^y)$, $C_2 = C_G(v_2^y)$, $Q_0 = O(C_2)$, $Q = O_{2'2}(C_2)$, $S_2 = S \cap C_2$, and $M_1 = M \cap C_1$ as well as $M_2 = M \cap C_2$.

Clearly, $O(M_1) = O(M_2) = O(M)$. Recall $(M!\,U) = 1771v_1 + 276v_2^y$; inspecting the structure of M_{24} we get $C_M(v_1)/O(M) \cong E(2^6) \cdot (\hat{3}\Sigma_6)$ and $M_2/O(M) \cong \mathrm{Aut}(M_{22})$.

Since $A_2 = O(A_2)\,A_c$ we may and do assume that q_0 is an element of $A_c - O(A_c)$ with $q_0{}^7 \in O(A_c) \leqslant O(A_2)$.

(3.3) The following statements hold:

(a) $S_2 = UE_2 \cdot (\langle r_1r_3d_1\,, r_1d_{1+}\,, t_1d_{2+}\rangle \times \langle r_1\rangle)$ is a Sylow 2-subgroup of C_2 with $Z(S_2) = \langle v_1\,, v_2{}^y\rangle$; moreover, $S_2 \in \mathrm{Syl}_2(UM_2)$.

(b) If Y is a subgroup of $T \cap S_2$ such that $|\,S_2{:}\ UY\,| = 2$ and $|\,Z(UY)| \geqslant 2^3$, then $Z(UY) = C_U(Y) = \langle v_1\,, v_1^{q_0{}^2}, v_2{}^y\rangle \simeq E(2^3)$.

Proof. (a) Clearly, $S_2 = UE_2 \cdot (\langle r_1r_3d_1\,, r_1d_{1+}\,, t_1d_{2+}\rangle \times \langle r_1\rangle)$ is a Sylow 2-subgroup of UM_2 and $Z(S_2) = \langle v_1\,, v_2{}^y\rangle$. As U is a characteristic subgroup of S_2, we have $S_2 \in \mathrm{Syl}_2(C_2)$.

(b) Let Y be a subgroup of $T \cap S_2$ such that $|\,S_2{:}\ UY\,| = 2$ and $|\,Z(UY)| \geqslant 2^3$ and let $R = \langle r_1\,, r_3d_1\,, d_{1+}\rangle$.

Clearly, $UY \lhd S_2 = U \cdot E_2 \cdot R \cdot \langle t_1d_{2+}\rangle$ and $Z(S_2) < C_U(Y) = Z(UY)$ as well as $E_2 \cap Y \geqslant E(2^3)$. If $E_2 \cap Y \simeq E(2^3)$, we got $C_U(E_2 \cap Y) = C_U(E_2) = C_V(E_2)$ and thus $C_U(Y) = C_U(S_2 \cap T) = Z(S_2)$ which is absurd. Hence, $E_2 \lhd Y$ and $C_U(Y) \leqslant C_V(E_2)$.

Let $Y_0 = Y \cap R\langle t_1d_{2+}\rangle$; then $Y = E_2 \cdot Y_0$ and $|\,Y_0\,| = 2^3$. As $C_U(E_2\,, r_1r_3d_1\,, r_1d_{1+}\,, t_1d_{2+}\rangle$; then $Y = E_2 \cdot Y_0$ and $|\,Y_0\,| = 2^3$. As $C_U(E_2\,, r_1r_3d_1\,, r_1d_{1+}\,, t_1d_{2+}) = Z(S_2)$ and as $\langle r_1r_3d_1\,, r_1d_{1+}\,, t_1d_{2+}\rangle$ is dihedral of order 8 with center $\langle r_1r_3d_1\rangle$, we have $r_1r_3d_1 \in Y_0$. So $C_U(Y) \leqslant C_V(E_2\,, r_1r_3d_1) = \langle v_1\,, v_1^{q_0{}^2}, v_2{}^y, v_1^{q_0q_2}\rangle$.

Now suppose that there exists an element x in Y_0 with $o(x) = 4$; then $x \in \langle r_1r_3d_1\,, r_1\rangle \cdot r_1d_{1+}r_1d_{2+}$. As $r_1r_3d_1 \in Y_0$, we may assume that $x \in \langle r_1\rangle \cdot d_{1+}t_1d_{2+}$. But in either case we get $C_U(Y) \leqslant C_V(E_2\,, r_1r_3d_1\,, x) = Z(S_2)$, a contradicition.

We have proved that $Y_0 \simeq E(2^3)$. Since $R\langle t_1d_{2+}\rangle \simeq D_8 \times Z_2$ with $Z(R\langle t_1d_{2+}\rangle) = \langle r_1\,, r_3d_1\rangle$, Y_0 contains r_1. So $C_U(Y) \leqslant C_V(E_2\,, r_1\,, r_3d_1) = \langle v_1\,, v_1^{q_0{}^2}, v_2{}^y\rangle$.

Q.E.D.

(3.4) (a) There are exactly three conjugacy classes of involutions in UM_2/U; these are represented by Ur_0, Ur_1, and Ur_1r_2. If $x \in \{r_0\,, r_1\,, r_1r_2\,, v_1^{q_0}r_1r_2\}$, then $\mathrm{ccl}_G(v_1) \cap U \cdot x = \mathrm{ccl}_U(x) = [U, x] \cdot x$; furthermore, $\mathrm{ccl}_G(v_2) \cap Ur_0 = \mathrm{ccl}_{U\langle q_0\rangle}(v_2{}^yr_0)$ contains exactly $7 \cdot 2^4$ elements.

(b) M_2 acts irreducibly on $U/\langle v_2{}^y\rangle$; moreover, $(M_2!\,U) = 3 \cdot 7 \cdot 11v_1 + 4 \cdot 5 \cdot 7 \cdot 11v_1^{q_0} + 1v_2{}^y + 3 \cdot 7 \cdot 11v_1v_2{}^y + 4 \cdot 11s$.

(c) $Q \in \{Q_0\langle v_2{}^y\rangle, Q_0 \cdot U\}$ and $C_1 = O(C_1)\,N_{C_2}(E) = O(C_1)\,UM_1$ with $M_1/O(M) \simeq E(2^5) \cdot \Sigma_5$.

Proof. (a) We have $D_0 = C_T(v_2{}^y) \cap A = \langle r_1r_3d_1\,, r_1d_{1+}\,, t_1d_{2+}\rangle \in \mathrm{Syl}_2((B^y)')$ and $D_0 \times \langle r_1\rangle \in \mathrm{Syl}_2(B^y)$, where $C_L(v_2{}^y) = O(L)\,E_2B^y$ with $B^y/O(B^y) \simeq \Sigma_6$ and $\langle B, y\rangle \leqslant A_c \leqslant N_G(U)$. Hence $UE_2B^y \leqslant UM_2$; as $UM_2/(O(M) \cdot U) \simeq$

$\mathrm{Aut}(M_{22})$, there are exactly three conjugacy classes of involutions in UM_2/U. Now we easily see that Ur_0, Ur_1, and Ur_1r_2 are representatives of those classes.

We have $C_U(r_0) \cong E(2^7)$, $\{x \in Ur_0 \mid x^2 = 1\} = C_U(r_0)\, r_0$, $C_U(r_0, q_0) = \langle v_1 v_1^{q_0 q_1} r_0 \rangle$ and $C(q_0) \cap C_U(r_0)\, r_0 = \{v_1 v_1^{q_0 q_1} r_0\}$. Moreover, $v_2^y r_0 \in Ur_0 \cap \mathrm{ccl}_G(v_2)$ and $\mathrm{ccl}_U(x) = [U, x] \cdot x$ for $x \in C_U(r_0)\, r_0$. As q_0 acts on both, $C_U(r_0)$ and $C_U(r_0)\, r_0$, we get $\mathrm{ccl}_G(v_1) \cap Ur_0 = [U, r_0]\, r_0 = \mathrm{ccl}_U(r_0)$ and $\mathrm{ccl}_G(v_2) \cap Ur_0 = \mathrm{ccl}_{U\langle q_0 \rangle}(v_2^y r_0)$.

Since $r_1r_2 \in \mathrm{ccl}_G(v_1)$, $v_1^{q_0} r_1 r_2 \in \mathrm{ccl}_2(v_2)$, and $C_U(r_1r_2) \cong E(2^6)$, we get $\mathrm{ccl}_G(v_{i+1}) \cap Ur_1r_2 = \mathrm{ccl}_U((v_1^{q_0})^i\, r_1r_2)$ for $i \in \{0, 1\}$.

As $C_U(r_1) \cong E(2^7)$, $Ur_1 \in \mathrm{ccl}_M(Ur_0)$. So the claimed results hold.

(b) Put $R_0 = \langle r_0, r_2, r_1r_3d_1, r_1d_{1+} \rangle$, $R = R_0 \times \langle r_1 \rangle$, and $\overline{N_G(U)} = N_G(U)/C_G(U)$. Recall that $\bar{M}_2 \cong \mathrm{Aut}(M_{22})$ and $C_{\bar{M}}(v_1) \cong E(2^6) \cdot (\hat{3}\Sigma_6)$ with $O_2(C_{\bar{M}}(v_1)) = \bar{R}_1 = \bar{E}$; moreover, $N_G(U) \cap C_1 = UM_1$ with $O(M_1) = O(M)$.

As $E_2\langle r_1, r_3d_1, d_+, a_1a_3, t_1d_{2+} \rangle = R\langle d_2, t_2, a_1a_3, t_1d_{2+} \rangle \leqslant C_L(v_1, v_2^y) \cap N_G(U)$ and $|\bar{M}_2 : \bar{M}_1| = |\mathrm{ccl}_{M_2}(v_1v_2^y)| \leqslant |\mathrm{ccl}_M(v_2^y)| - 1 = 275$, we get $2^8 \cdot 3 \cdot 5 \mid |\bar{M}_1| \mid 2^8 \cdot 3^2 \cdot 5$.

Now suppose that there exists a subgroup X of $\bar{M}_1$ with $|X| = 3^2$. Since $X \in \mathrm{Syl}_3(\bar{M}_2)$ and $\Sigma_6 \cong \overline{B^y} \leqslant \bar{M}_2 \cap \bar{A}_c$ as well as $\langle \bar{a}_1, \bar{a}_3 \rangle \in \mathrm{Syl}_3(\bar{B})$, there exists an element m of $\bar{M}$ with $X^m = \langle \bar{a}_1, \bar{a}_3 \rangle$. We have $C_U(a_1, a_3) = \langle v_1, v_2, s \rangle$, $v_1\langle v_2, s \rangle \subseteq \mathrm{ccl}_G(v_1)$, and $\{v_2, v_2s, s\} \subseteq \mathrm{ccl}_G(v_2)$. As $v_2^y \cdot \langle v_1 \rangle \subseteq \mathrm{ccl}_G(v_2)$ and $\langle v_1, v_2^y \rangle^m \leqslant \langle v_1, v_2, s \rangle$, we get $v_1^m \in \langle v_2, s \rangle^\# \subseteq \mathrm{ccl}_G(v_2)$, a contradiction.

We have proved that $|\bar{M}_1| = 2^8 \cdot 3 \cdot 5$ and so $|\mathrm{ccl}_{M_2}(v_1)| = |\mathrm{ccl}_{M_2}(v_1v_2^y)| = 3 \cdot 7 \cdot 11$. Inspecting the structure of M_{22} and $\mathrm{Aut}(M_{22})$ we see that $(\bar{M}_2)' \cap \bar{M}_1 \cong E(2^4) \cdot \Sigma_5$ and $\bar{M}_1 \cong E(2^5) \cdot \Sigma_5$.

Obviously, $v_1^{q_0} \in \mathrm{ccl}_M(v_1) - \mathrm{ccl}_{M_2}(v_1)$ and $s \in \mathrm{ccl}_M(v_2) - (\mathrm{ccl}_{M_2}(v_1v_2^y) \cup \{v_2^y\})$. Since $C_{\bar{M}}(v_1^{q_0}) \cong E(2^6) \cdot (\hat{3}\Sigma_6)$ and $Z(S_2) = \langle v_1, v_2^y \rangle$, $|C_{\bar{M}_2}(v_1^{q_0})|$ divides $2^7 \cdot 3^2 \cdot 5$ and $|C_{\bar{M}_2}(s)|$ divides $2^7 \cdot 3^2 \cdot 5 \cdot 7$.

Let $x \in C_{A_c}(v_2^y, s) = O(A_c)\,(B^y)'$ with $o(\bar{x}) = 5$. Then $C_U(x) = C_V(E_2, x)\,\langle s \rangle \cong E(2^3)$ and $\{s\} \cup C_V(E_2, x)^\# \subseteq \mathrm{ccl}_G(v_2)$. Moreover, there exists an element d in $C_{A_c}(x \bmod O(A_c))$ with $o(\bar{d}) = 3$. Since $Z_{15} \not\lesssim \mathrm{Aut}(M_{22})$ and $C_{\bar{M}}(u) \cong \mathrm{Aut}(M_{22})$ for each u in $\mathrm{ccl}_M(v_2)$, we get $|\mathrm{ccl}_M(v_2) \cap C_U(x)| = 6$ and therefore $|\mathrm{ccl}_M(v_1) \cap C_U(x)| = 1$. Hence $5 \nmid |C_{\bar{M}_2}(v_1^{q_0})|$.

So $2 \cdot 11 \mid |\mathrm{ccl}_{M_2}(s)| \leqslant 276 - (3 \cdot 7 \cdot 11 + 1) = 4 \cdot 11$ and $2 \cdot 5 \cdot 7 \cdot 11 \mid |\mathrm{ccl}_{M_2}(v_1^{q_0})| \leqslant 1771 - 231 = 4 \cdot 5 \cdot 7 \cdot 11$.

Now let $x \in \{v_1^{q_0}, s\}$; note that $\{x, v_2^y x\} \subseteq \mathrm{ccl}_M(x)$. If $|C_{M_2}(x)|_2 = 2^7$, then, by (3.3) (b), there exists an element m in M with $\langle v_2^y, x \rangle^m \leqslant \langle v_1, v_1^{q_0^2}, v_2^y \rangle$. But this is impossible, because $\langle v_1, v_1^{q_0^2} \rangle^\# \subseteq \mathrm{ccl}_M(v_1)$ and $\langle v_1, v_1^{q_0^2} \rangle \cdot v_2^y \subseteq \mathrm{ccl}_M(v_2)$. Therefore $|C_{M_2}(x)|_2 = 2^6$ and $4 \mid |\mathrm{ccl}_{M_2}(x)|$. This finally proves $(M_2!\, U) = 3 \cdot 7 \cdot 11 v_1 + 4 \cdot 5 \cdot 7 \cdot 11 v_1^{q_0} + 1 v_2^y + 3 \cdot 7 \cdot 11 v_1v_2^y + 4 \cdot 11 s$.

(c) Obviously $Q \in \{Q_0\langle v_2^y \rangle, Q_0 \cdot U\}$, because M_2 acts irreducibly on $U/\langle v_2^y \rangle$.

Since $C_G(v_1) = O(C_G(v_1))\, N_G(E)$, we easily see that $C_1 = (C_2 \cap O(C_G(v_1))) \times$

$N_{C_2}(E) = O(C_1)\, N_{C_2}(E)$ and $O(C_1) \cap N_{C_2}(E) = O(N_{C_2}(E)) = O(C_G(E))$. Hence $C_1/O(C_1) \cong N_{C_2}(E)/O(C_G(E))$ and so, by (3.1), $|\, C_1/O(C_1)| = 2^{19} \cdot 3 \cdot 5$.

Note that $UM_1 \leqslant C_1$ and $UM_1/O(M) \cong E(2^{11}) \cdot E(2^5) \cdot \Sigma_5$. Hence, $2^{19} \cdot 3 \cdot 5 \mid |\, O(C_1)\, UM_1/O(C_1)| \leqslant |\, C_1/O(C_1)| = 2^{19} \cdot 3 \cdot 5$. So $O(C_1) \cap UM_1 = O(M)$ and $C_1 = O(C_1)\, UM_1$. Q.E.D.

(3.5) We have $C_2 = O(C_2)\, UM_2$ with $O(C_2) \cap UM_2 = O(M)$ and $C_2/O(C_2) \cong E(2^{11}) \cdot \mathrm{Aut}(M_{22})$. In particular, the group C_2 is 2-constrained.

Proof. Put $D_0 = \langle r_1 r_3 d_1, r_1 d_{1+}, t_1 d_{2+}\rangle$, $D_1 = \langle d_2, t_2, t_1, d_{2+}\rangle$, $R = \langle r_0, r_2, r_1 r_3 d_1, r_1 d_{1+}\rangle$, and $S_2^* = U \cdot E_2 \cdot D_0 = U \cdot R \cdot D_1$. Recall that $C_L(v_2{}^y) = O(L)\, E_2 B^y$ with $B^y/O(B^y) \cong \Sigma_6$, $D_0 \times \langle r_1\rangle \in \mathrm{Syl}_2(B^y)$, and $D_0 \in \mathrm{Syl}_2((B^y)')$; moreover, $S_2 = S_2^* \cdot \langle r_1\rangle \in \mathrm{Syl}_2(UM_2) \subseteq \mathrm{Syl}_2(C_2)$.

Suppose that C_2 contains no subgroups with index 2. Then there exists an element g in C_2 such that $r_1{}^g \in S_2^*$. As $\{r_0, r_1, v_2{}^y r_1\} \subseteq \mathrm{ccl}_G(v_1)$, $\{v_2{}^y r_0, v_1 v_2{}^y\} \subseteq \mathrm{ccl}_G(v_2)$, and $\mathrm{ccl}_G(v_1) \cap Ur_0 = \mathrm{ccl}_U(r_0)$, we conclude that $r_1{}^g \in \mathrm{ccl}_{M_2}(v_1^{q_0})$. So we may assume that $r_1{}^g = v_1^{q_0}$. Let X be an S_2-subgroup of $C_{C_2}(v_1^{q_0})$ containing $S_2 \cap C(v_1^{q_0})$; since $X \leqslant UM_2$ and $4 \mid |\, \mathrm{ccl}_{UM_2}(v^{q_0})|$, we get $X = S_2 \cap C(v_1^{q_0}) = UE_2\langle d_{1+}, r_1\rangle$ and $Z(X) = \langle v_1, v_2{}^y, v_1^{q_0}\rangle$. Note that $C_{S_2}(r_1) = C_U(r_1) \cdot R\langle d_2, t_1 d_{2+}, r_1\rangle$ and $C_S(C_U(r_1)) = U\langle r_1\rangle$. Hence, $Z(C_{S_2}(r_1)) = \langle v_1, v_2{}^y, r_1\rangle$ is also the center of an S_2-subgroup of $C_{C_2}(r_1)$. So we may assume that $\langle v_1, v_2{}^y, r_1\rangle^g = \langle v_1, v_2{}^y, v_1^{q_0}\rangle$. Since $\langle v_1, v_2{}^y, v_1^{q_0}\rangle \cap \mathrm{ccl}_G(v_2) = v_2{}^y \cdot \langle v_1\rangle$ and $v_2^{yg} = v_2{}^y$, we get $(v_1 v_2{}^y)^g = v_1 v_2{}^y$. Thus $g \in C_1 = O(C_1)\, UM_1$. Therefore $(v_1^{q_0})^{g^{-1}} = r_1 \in O(C_1)\, U$; but this is impossible.

Hence, C_2 contains a subgroup C_2^* with $|\, C_2 : C_2^*\,| = 2$ and $S_2^* \in \mathrm{Syl}_2(C_2^*)$. Clearly, U is weakly closed in S_2^* w.r.t. C_2^*.

In the following we assume by way of contradiction that U is not strongly closed in S_2^* w.r.t. C_2^*.

So we put $\beta = \{X \mid X \leqslant S_2^*, X \nleqslant U, \exists\, g \in C_2^*: X^g \leqslant U\}$ and $r = \max\{m(X/X \cap U) \mid X \in \beta\}$. We know that S_2^*/U is isomorphic to S_2-subgroups of the Mathieu group M_{22}; hence $r \leqslant 4$. As M_{22} contains exactly one conjugacy class of involutions and $[U, r_0] \cong E(2^4)$, application of the results in [6, Chap. 9] yields $r = 4$; furthermore, β contains elements of order divisible by 2^7.

Let X be an element of maximal order of β and let t be an element of $X - U$; put $X_1 = (X \cap U) \cdot \langle t\rangle$ and $X_2 = N_U(X_1)$. Then $|\, X_2\,| = |\, C_U(t)| \cdot |[U, t] \cap X \cap U\,|$ and $X_1 \in \beta$; in particular there exists an element c in C_2^* such that $X_1{}^c \leqslant U$ and $X_2{}^c \leqslant S_2^*$. If $|\, X_2\,| > |\, X\,|$, we get $X_2{}^c \leqslant U$ because of the maximality of $|\, X\,|$; hence, $X_1 X_2 \in \beta$ and $|\, X_2\,| \leqslant |\, X_1 X_2\,| \leqslant |\, X\,|$, but this is absurd. So $|\, X\,| \geqslant |\, X_2\,|$. Clearly, $X_0 = [U, t]\,(X \cap U) \leqslant C_U(t)$; thus $|\, X\,| \geqslant |\, C_U(t)| \cdot |[U, t]| \cdot |\, X \cap U\,| / |\, X_0\,| \geqslant |[U, t]| \cdot |\, X \cap U\,| = 2^4 \cdot |\, X \cap U\,|$. As $|\, X/X \cap U\,| \leqslant 2^r = 2^4$, we have the following:

$$2^7 \mid |\, X\,| = 2^4 \cdot |\, X \cap U\,| \qquad \text{and} \qquad C_U(t) = [U, t]\,(X \cap U) \cong E(2^7);$$

in particular, $UX \in \{UE_2, UR\}$.

Suppose $UX = UR$; then $E(2^3) \lesssim X \cap U \leqslant C_U(R) = \langle v_1, v_2{}^y, v_2^{ya_1} \rangle$. Note that $[U, r_0] = \langle [V, r_0], [s, r_0] \rangle = \langle v_1, v_1^{q_0}, v_1^{q_0{}^2}, v_1^{q_0 a_1} r_0 \rangle$. Putting $t = r_0$ we get $E(2^7) \cong C_U(r_0) = [U, r_0]\, C_U(R) \cong E(2^6)$ which is absurd. This contradiction yields $UX = UE_2$ and $X \cap U \leqslant C_U(E_2) = C_V(E_2)$.

Let $(e_1, e_2, e_3, e_4) = (r_0, r_2, d_2, t_2)$. Then there exist elements u_i in $C_U(e_i)$ with $u_i e_i \in X$ and $[u_i, e_j] = [u_j, e_i]$ for all $i, j \in \{1, 2, 3, 4\}$. Copying the arguments of (2.4) (d) we get $X \leqslant W^* E_2 = VE_2$ and $u_i \in C_{W^*}(e_i) = C_V(E_2)\, \langle e_i e_i{}^s \rangle$ for $i \in \{1, 2, 3, 4\}$.

If $[u_i, e_j] = 1$ for some $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$, then $u_k \in C_V(E_2)$ for all $k \in \{1, 2, 3, 4\}$; hence $X \leqslant C_V(E_2)\, E_2 = W$. So suppose $[u_i, e_j] = [u_j, e_i] \neq 1$ for all $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$. Then $u_i \in C_V(E_2) \cdot e_i e_i{}^s$ and so $u_i e_i \in C_V(E_2) \cdot e_i{}^s \subseteq V$ for all $i \in \{1, 2, 3, 4\}$; hence $X \leqslant V$. Since $s \in C_2^*$ with $W^s = V$, we may and do assume that $X \leqslant V$.

If $|X|$ is divisible by 2^9, then $2^9 - 156 = 356 \leqslant |\operatorname{ccl}_G(v_2) \cap X| \leqslant |\operatorname{ccl}_G(v_2) \cap U| = 276$; but this is impossible.

So suppose that $|X| = 2^8$; then $E(2^4) \cong X/(X \cap U) \cong X \cap U < C_V(E_2)$ and $\langle v_1, v_1^{q_0{}^2}, v_1^{q_2} v_1^{q_0 q_2} \rangle \cap (X \cap U) \neq 1$. Note that $\{v_1^{q_0{}^2}, v_1^{q_2} v_1^{q_0 q_2}\} \langle v_1 \rangle \cup \{v_1\} \subseteq \operatorname{ccl}_{M_2}(v_1)$ and $v_1^{q_0{}^2} v_1^{q_2} v_1^{q_0 q_2} \langle v_1 \rangle \subseteq \operatorname{ccl}_{M_2}(v_1 v_2{}^y)$; thus $X \cap U \cap \operatorname{ccl}_{M_2}(v_1) \neq \varnothing$.

Let $w \in X \cap U \cap \operatorname{ccl}_{M_2}(v_1)$. We know that there exists an element g of C_2 such that $X^g \leqslant U$. Therefore $w^g \in \operatorname{ccl}_{M_2}(v_1)$ and we may assume that $w^g = w$. So M_2 contains an element m such that $v_1{}^m = w$ and $g \in C_G(v_2{}^y, w) = O(C_1)^m\, UM_1{}^m$. Since $X \not\leqslant U$ and $UM_1{}^m \leqslant N_G(U)$, we may assume that $g \in O(C_1)^m$. Thus, $[X, g] \leqslant UE_2 \cap O(C_1)^m = 1$. This implies $X = X^g \leqslant U$, a contradiction.

We have proved that $|X| = 2^7$. Hence, $E(2^3) \cong X \cap U < C_V(E_2)$ and $C_U(t) = C_{W^*}(t) = [U, t] \times (X \cap U)$ for any $t \in X - U$. As $UX = UE_2$, we get $X \cap U \cap [V, e] \leqslant X \cap U \cap [U, e] = 1$ for all $e \in E_2$. Since E_2 is normal in T, we have $X \cap U \cap \operatorname{ccl}_T(w) = \varnothing$ for any $e \in E_2$ and $w \in [V, e]^{\#}$. So, by (1.9), $(X \cap U) \subseteq \operatorname{ccl}_T(v_2{}^y) \cup \operatorname{ccl}_T(v_2^{ya_1}) \cup \operatorname{ccl}_T(v_2)$.

Note that $\operatorname{ccl}_T(v_2{}^y) = v_2{}^y \langle v_1, v_1^{q_0{}^2} \rangle$ and $\operatorname{ccl}_T(v_2^{ya_1}) = v_2^{ya_1}\{\langle v_1, v_2{}^y \rangle \cup \langle v_1 \rangle \times \{v_1^{q_0}, v_1^{q_0{}^2} v_1^{q_2}\}\}$ as well as $v_2{}^y \cdot v_2^{ya_1} \langle v_1 \rangle \{v_1^{q_0}, v_1^{q_0{}^2} v_1^{q_2}\} = v_1^{q_0 q_2} \langle v_1 \rangle \{v_1^{q_0} v_1^{q_0{}^2}, v_1^{q_2}\} \subseteq \operatorname{ccl}_T(v_1^{q_0 q_2})$. Hence, there exists an element $w_0 \in v_2^{ya_1} \cdot \langle v_1 \rangle$ such that $\langle v_2{}^y, w_0 \rangle \leqslant X \cap U \leqslant \langle v_2{}^y, w_0 \rangle \cup \operatorname{ccl}_T(v_2)$.

Put $I = v_2\{1, v_1^{q_0{}^2}, v_1^{q_0} v_1^{q_0 q_2}, v_1 v_1^{q_0{}^2} v_1^{q_0} v_1^{q_0 q_2}\}$; then $I \cup v_2{}^y \cdot I = \{x \in \operatorname{ccl}_T(v_2) \mid v_2{}^y x \in \operatorname{ccl}_T(v_2)\}$ and $|X \cap U \cap I| = |X \cap U \cap v_2{}^y \cdot I| = 2$. Moreover, if $X \cap U \cap I = \{i_1, i_2\}$, then $i_1 i_2 \in C_V(E_2) \cap \operatorname{ccl}_G(v_2)$. Making use of (1.10) we get $\{i_1, i_2\} \in \{\{v_2, v_2 v_1^{q_0} v_1^{q_0 q_2}\}, \{v_2 v_1^{q_0{}^2}, v_2 v_1 v_1^{q_0{}^2} v_1^{q_0} v_1^{q_0 q_2}\}\}$ and $i_1 i_2 \in v_1^{q_0} v_1^{q_0 q_2} \langle v_1 \rangle$. As $v_2{}^y \in X \cap U$, we have $\varnothing \neq X \cap U \cap v_2{}^y v_1^{q_0} v_1^{q_0 q_2} \cdot \langle v_1 \rangle \subseteq v_1^{q_2} v_1^{q_0 q_2} \cdot \langle v_1 \rangle \subseteq \operatorname{ccl}_T(v_1^{q_0 q_2})$. But this contradicts the fact that $(X \cap U)^{\#} \subseteq \operatorname{ccl}_G(v_2)$.

This contradiction finally proves that U is strongly closed in S_2^* w.r.t. C_2^*. So C_2^* is an $S^*(U)$-group in the sense of [6].

Put $H = U^{C_2^*}$. As S_2^* splits over U, we easily see that $U \in \operatorname{Syl}_2(H)$. Since $H \cap N_G(U) \leqslant UM_2$, we have $U \leqslant H \cap N_G(U) \leqslant O(M) \cdot U = C_G(U)$. So

$H = O(H) \cdot U$. As H is normal in C_2^*, the Frattini argument yields $C_2^* = O(H)\, N_{C_2^*}(U)$. Hence $Q = O_{2',2}(C_2^*) = Q_0 \cdot U$ and $C_2 = O(C_2)\, N_{C_2}(U) = O(C_2)\, UM_2$ with $O(M) \leqslant O(C_2)$.

Obviously, C_2 is 2-constrained. Q.E.D.

We know that v_1 and $v_2{}^y$ are representatives of the two conjugacy classes of involutions of G and that both, $C_G(v_1)$ and $C_G(v_2{}^y)$ are 2-constrained.

Since $G/O(G)$ satisfies assumptions (A1), (A2), and (A3), we may assume that $O(G) = 1$. Now application of a theorem of [18] yields $O(N_G(U)) \leqslant O(C_G(v_2{}^y)) = 1$ and $O(C_G(v_1)) = 1$. So, by the main result of [15], $G \cong J_4$.

We have proved the following theorem.

THEOREM 3. *Let G be a finite group satisfying the assumptions* (A1), (A2), *and* (A3). *Then $G/O(G)$ is isomorphic to J_4.*

Furthermore, the proof of the Main Theorem is complete.

REFERENCES

1. M. ASCHBACHER AND G. SEITZ, Involutions in Chevalley groups over fields of even order, *Nagoya Math. J.* **63** (1976), 1–91.
2. G. BELL, Cohomology of degree 0, 1, and 2 of $SL_n(2)$, to appear.
3. R. BRAUER, On groups of even order with Abelian 2-Sylow subgroups, *Arch. Math. (Basel)* **13** (1962), 55–60.
4. B. DEXHEIMER, "Eine Kennzeichnung von $D_5(2)$ durch eine 2-lokale Untergruppe," Diplomarbeit, Universität Mainz, 1978.
5. W. GASCHÜTZ, Zur Erweiterungstheorie der endlichen Gruppen, *J. Reine Angew. Math.* **190** (1952), 93–107.
6. D. GOLDSCHMIDT, 2-Fusion in finite groups, *Ann. of Math.* **99** (1974), 70–117.
7. D. GORENSTEIN, "Finite Groups," Harper & Row, New York, 1968.
8. D. GORENSTEIN AND K. HARADA, Finite groups whose 2-subgroups are generated by at most four elements, *Mem. Amer. Math. Soc.* **147** (1974).
9. M. HALL, Simple groups of order less than one million, *J. Algebra* **20** (1972), 98–102.
10. K. HARADA, On finite groups having self-centralizing 2-subgroups of small order, *J. Algebra* **33** (1975), 144–160.
11. K. HARADA, Finite groups having 2-local subgroups $E_{16} \cdot L_4(2)$, to appear.